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MATHEMATICS

On Proper Isomorphisms of (μ, Δ, γ)-Systems. I.

by

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1. Introduction. According to [2], a (μ, Δ, γ)-system,

$$H[\{H^n\}, \{H^n(2)\}, \mu, \Delta, \gamma],$$

is a set of Abelian groups H^n, $H^n(2)$, $n=1,2,\ldots$, and homomorphisms

$$\mu : H^n \to H^n(2),$$

$$\Delta : H^n(2) \to {}_2H^{n+1},$$

$$\gamma_p^n : H^n(2) \to H^{n+p}(2),$$

where p is even *) and ${}_2H^{n+1}$ is the subgroup of H^{n+1} consisting of elements g such that $2g=0$, subject to the conditions that $\Delta^{-1}(0)=\mu H^n$ and that Δ has a right inverse Δ^*.

A homomorphism h of the (μ, Δ, γ)-system H into the (μ, Δ, γ)-system $\bar{H}=[\{\bar{H}^n\}, \{\bar{H}^n(2)\}, \bar{\mu}, \bar{\Delta}, \bar{\gamma}]$ is a set of homomorphisms $h : H^n \to \bar{H}^n$, $h : \bar{H}^n(2) \to \bar{H}^n(2)$, $n=1,2,\ldots$, subject to $\bar{\mu}h=h\mu$, $\bar{\Delta}h=h\Delta$, $\bar{\gamma}h=h\gamma$.

If K is a finite simplicial complex, then we may form a (μ, Δ, γ)-system by taking $H^n=H^n(K)$, $H^n(2)=H^n(K, Z_2)$, where Z_2 is the group of integers reduced mod 2, μ the natural map, $\Delta=S_q 1=\frac{1}{2}\delta$, and γ_p^n Steenrod square **). A homotopy equivalence $f : L \to K$ induces an isomorphism $f^* : H(K) \to H(L)$, so that the invariants of isomorphism classes of (μ, Δ, γ)-systems give rise to invariants of homotopy type.

Among such invariants we have already introduced the block invariants of the homomorphisms γ_p^n. However, as remarked in [2], it is not in general possible to reduce simultaneously the matrices of every γ_p^n

*) According to [4], p may be confined to be powers of 2. We do not assume that a homomorphism γ_p^n is given for every value of (n,p).

**) We could extend the definition of (μ, Δ, γ)-systems to include more groups and more homomorphisms among its groups.

to normal form. We consider in this note the simplest case in which simultaneous reduction is impossible and obtain new numerical invariants, which we call relative block invariants, which, together with the block invariants and classical invariants, completely determine the isomorphisms class of this restricted type of (μ, Δ, γ)-systems.

2. Let H and $\bar{H}$ denote two (μ, Δ, γ)-systems. If there is a system of isomorphisms $h^n : H^n \to \bar{H}^n$ $(n = 1, 2, \ldots,)$, then, according to J. H. C. Whitehead [3], $\{h^n\}$ may be extended to a (μ, Δ)-isomorphism $h : H \to \bar{H}$ which induces, according to [1], a new (μ, Δ, γ)-system

$$H^* = H^*[\{H^n\}, \{\dot{H}^n(2)\}, \mu, \Delta, h^{-1}\gamma h].$$

We will say that an isomorphism of the systems H into the system H' is an automorphism if H, H' coincide as (μ, Δ)-systems.

LEMMA 1. *Let $f : H \to \bar{H}$ be a (μ, Δ)-isomorphism. Then f is an isomorphism if, and only if, $h^{-1}f : H \to H^*$ is an automorphism.*

This is a trivial extension of an argument in [1].

Let E be a set of pairs of integers (n, p), where p is even. If we know *) a set A, of invariants of an isomorphism class of (μ, Δ, γ)-systems, such that these invariants determine the normal forms N_p^n of γ_p^n in case $(n, p) \in E$, there must be an automorphism g carrying the (μ, Δ, γ)-system H into another, namely gH; in the latter γ_p^n becomes N_p^n if $(n, p) \in E$. In other words, we have

$$g\gamma_p^n g^{-1} = N_p^n \qquad \text{if} \qquad (n, p) \in E.$$

If the isomorphism class containing H^* has the same set of invariants A as the class containing H, then there is an automorphism g^* of H^* such that

$$g^*\gamma_p^n g^{*-1} = N_p^n, \qquad \text{if} \qquad (n, p) \in E.$$

Then we have

LEMMA 2. *The (μ, Δ, γ)-systems H and H^* are automorphic, if and only if, gH and g^*H^* are automorphic.*

Hence, our algebraic problem is to find the complete set of numerical invariants of $\{\gamma_p^n\}$ $((n, p) \in E)$ under all the (μ, Δ)-automorphisms h, which satisfy the following conditions

$$h\gamma_p^n = \gamma_p^n h, \qquad (n, p) \in E;$$

$$h\gamma_p^n = \gamma_p^{*n} h, \qquad (n, p) \notin E.$$

*) For example, we know block invariants which can, at least, help us to determine the normal form of one γ_p^n in H.

Of course we must investigate the necessary and sufficient conditions, for a system of automorphisms of $\{H^n(m)\}$ $(m=0,2;\ n=1,2,\ldots)$ to constitute a (μ,Δ)-automorphism.

By the use of the above argument we solve a particular problem as follows: By $\mathfrak{L}$ we mean the set of (μ,Δ,γ)-systems, each of which consists of no more than two non-trivial $\gamma's$, namely γ_p^n and γ_p^m, with the additional condition that between the sets of groups $(H^n,H^{n+1},H^{n+p},H^{n+p+1})$ and $(H^m,H^{m+1},H^{m+k},H^{m+k+1})$ there exists one and only one common group. Then $\mathfrak{L}$ is the type of (μ,Δ,γ)-system mentioned in the Introduction.

We determine the complete set of numerical invariants of isomorphism classes of (μ,Δ,γ)-systems of $\mathfrak{L}$ and reach the normal forms of such classes obtaining some essentially new invariants. With respect to a given complex we may, in general, choose a pair of Steenrod squares satisfying our condition so that these squares, together with the cohomology groups $H^n(m)$ $(m=0,2;\ n=1,2,\ldots)$ and associated homomorphisms, constitute a system of $\mathfrak{L}$. Consequently we obtain new numerical invariants of the complex for each such choice of pairs of Steenrod squares.

Let H be a system belonging to $\mathfrak{L}$. Then there are four cases as follows:

(a) $n=m+1,\ k>2,\ p\neq k,\ k-2$ or $m=n+1,\ p>2,\ k\neq p,\ p-2;$

(b) $n+p=m+k+1,\ p>2,\ p\neq k,\ k+2;$ or $m+k=n+p+1,\ k>2,\ k\neq p,\ p+2;$

(c) $n=m+k+1$ or $m=p+n+1;$

(d) $n+1=m+k,\ k>2,\ p>2;$ or $m+1=n+p,\ p>2,\ k>2.$

Let γ_p^n be reduced to normal form, N_p^n, by the method in [2]. Since H^n is an Abelian group, we consider H^n as the direct sum of cyclic groups of prime power order. Let $u^n(1),\ldots,u^n(s_1^n),\ldots,u^n(s_2^n),\ldots,u^n(s_{N(n)+1}^n)$ be generators of H^n such that $u^n(s_{i-1}^n+a)$, $a=1,\ldots,s_i^n-s_{i-1}^n$ $(s_0^n=0)$, is of order m_i^n, a power of 2, where $m_i^n<m_{i+1}^n$ and $m_{N(n)+1}^n=\infty$. By $\tau_{i,j}^{(n,p)}$ $(i=1,\ldots,N(n)+N(n+1)+1;\ j=1,\ldots,N(n+p)+N(n+p+1)+1)$ we mean the block invariants (see [2]) of γ_p^n.

For convenience we put

$$\tau_{i,N(n)+N(n+1)+2}^{(n,p)}=s_i^n-s_{i-1}^n-\sum_{j=1}^{N(n+p)+N(n+p+1)+1}\tau_{i,j}^{(n,p)},\qquad i=1,\ldots,1+N(n);$$

$$\tau_{i,N(n)+N(n+1)+2}^{(n,p)}=s_{N(n+1)+N(n)+2-i}^{n+1}-s_{N(n+1)+N(n)+1-i}^{n+1}-\sum_{j=i}^{N(n+p)+N(n+p+1)+1}\tau_{i,j}^{(n,p)},$$

$$i=2+N(n),\ldots,N(n)+N(n+1)+1;$$

$$\tau^{(n,p)}_{N(n+p)+N(n+p+1)+2,j}=s^{n+p+1}_{j}-s^{n+p+1}_{j-1}-\sum_{i=1}^{N(n)+N(n+1)+1}\tau^{(n,p)}_{i,j},\quad j=1,\dots,N(n+p+1);$$

$$\tau^{(n,p)}_{N(n+p)+N(n+p+1)+2,j}=s^{n+p}_{N(n+p)+2-j+N(n+p+1)}-s^{n+p}_{N(n+p)+N(n+p+1)+1-j}-\sum_{i=1}^{N(n)+N(n+1)+1}\tau^{(n,p)}_{i,i}$$

$$j=N(n+p+1)+1,\dots,N(n+p)+N(n+p+1)+1.$$

In case (a) we define

(1) $$\alpha=N(m+k)+N(m+k+1)+1,$$

(2)
$$a_i=s^{m+k+1}_{i}-s^{m+k+1}_{i-1},\qquad i=1,\dots,N(m+k+1);$$
$$a_i=s^{m+k}_{N(m+k)+N(m+k+1)+2-i}-s^{m+k}_{N(m+k)+N(m+k+1)+1-i},$$
$$i=N(m+k+1)+1,\dots,N(m+k)+N(m+k+1)+1;$$

(3) $$\beta=N(m)+1+N(m+1)(N(n+p)+N(n+p+1)+2);$$

(4)
$$b_j=s^{m}_{j}-s^{m}_{j-1},\quad j=1,\dots,N(m)+1;$$
$$b_j=\tau^{(n,p)}_{N(n)-l+1,l(N(n+p)+N(n+p+1)+2)+N(m)-j+2},$$
$$j=N(m)+2+(l-1)\big(N(n+p)+N(n+p+1)+2\big),\dots,$$
$$N(m)+1+l\big(N(n+p)+N(n+p+1)+2\big);$$
$$l=1,\dots,N(m+1).$$

In case (b) we replace (1), (2), (3), (4) by (5), (6), (7), (8) as follows:

(5) $$\alpha=N(m+k)+1+N(m+k+1)\big(N(n+1)+N(n)+2\big),$$

(6)
$$a_i=\tau^{(n,p)}_{i(N(n+1)+N(n)+2)-i+1,N(n+p+1)+N(n+p)+2-l},$$
$$i=(l-1)\big(N(n+1)+N(n)+2\big)+1,\dots,l\big(N(n+1)+N(n)+2\big);$$
$$l=1,\dots,N(m+k+1);$$
$$a_i=s^{m+k}_{N(m+k)+2-i+N(m+k+1)(N(n+1)+N(n)+2)}-s^{m+k}_{N(m+k)+1-i+N(m+k+1)(N(n+1)+N(n)+1)},$$
$$i=N(m+k+1)\big(N(n+1)+N(n)+2\big)+1,\dots,\alpha;$$

(7) $$\beta=N(m)+N(m+1)+1,$$

(8)
$$b_j=s^{m}_{j}-s^{m}_{j-1},\qquad j=1,\dots,N(m)+1;$$
$$b_j=s^{m+1}_{N(m+1)+N(m)+2-j}-s^{m+1}_{N(m+1)+N(m)+1-j},\qquad j=1,\dots,\beta.$$

In case (c), β and b_j $(j=1,\dots,\beta)$ are still given by (7) and (8), but α and $a_i (i=1,\dots,\alpha)$ are now given by

(9) $$\alpha=N(m+k)+1+N(n)\big(N(n+p)+N(n+p+1)+2\big)$$

and

$$a_i=\tau^{(n,p)}_{i-(l-1)(N(n+p)+N(n+p+1)+2),l}$$

$$i=(l-1)\big(N(n+p)+N(n+p+1)+2\big)+1,\dots,l\big(N(n+p)+N(n+p+1)+2\big);$$

$$l=1,\dots,N(n);$$

$$(10)\qquad a_i=s^{m+k}_{N(m+k)+2-i+N(n)(N(n+p)+N(n+p+1)+2)}-s^{m+k}_{N(m+k)+1-i+N(n)(N(n+p)+N(n+p+1)+2)},$$

$$i=N(n)\big(N(n+p)+N(n+p+1)+2\big)+1,\dots,\alpha.$$

In case (d) β and b_j are again given by (7) and (8), but we need to replace (9) and (10) by

$$(11)\qquad \alpha=N(m+k+1)+1+N(n+1)\big(N(n+p)+N(n+p+1)+2\big),$$

and

$$a_i=s^{m+k+1}_i-s^{m+k+1}_{i-1},\qquad i=1,\dots,N(m+k+1);$$

$$(12)\qquad a_{N(m+k+1)+1}=s^{m+k}_{N(m+k)+1}-s^{m+k}_{N(m+k)},$$

$$a_i=\tau^{(n,p)}_{l,i-N(m+k+1)-1-(l-1)(N(n+p)+N(n+p+1)+2)},$$

$$i=N(m+k+1)+2+(l-1)\big(N(n+p)+N(n+p+1)+2\big),\dots,$$

$$N(m+k+1)+1+l\big(N(n+p)+N(n+p+1)+2\big);$$

$$l=1,\dots,N(n+1).$$

Now suppose that γ^n_p has been normalised. Then γ^m_k of H is represented by a matrix $[\gamma^m_k]$ of $\sum_{j=1}^{\beta} b_j$ rows and $\sum_{i=1}^{\alpha} a_i$ columns. The first $\sum_{j=1}^{n} b_j$ rows and first $\sum_{i=1}^{s} a_i$ columns of $[\gamma^m_k]$ intersect in a matrix of rank $T_{n,s}$ say. We define, as in [1],

$$\tau_{n,s}=T_{n,s}-T_{n-1,s}-T_{n,s-1}+T_{n-1,s-1},$$

and call the numbers $\sigma_{n,s}$ the relative block invariants of $[\gamma^m_k]$ with respect to $[N^n_p]$. We have the following

THEOREM. *Among (μ,Δ,γ)-systems of Ω each isomorphism class is completely determined by Betti numbers, torsions, block invariants and relative block invariants.*

The normal form of each isomorphism class of (μ,Δ,γ)-systems of Ω may be given, according to the method given in [1] or [2], by two matrices $[N^n_p]$ and $[N^m_k]$, where $[N^n_p]$ has $\big(N(n+p)+N(n+p+1)+1\big)\big(N(n)+N(n+1)+1\big)$ blocks $B_{u,v}$, and $[N^m_k]$ has $\alpha\beta$ blocks $\bar{B}_{n,s}$, such that

in each column and each row there is only one non-zero element *) and that the number of non-zero elements in each block of $[N_p^n]$ (or $[N_k^m]$) is equal to the corresponding (relative) block invariant of the given block.

Moreover, the normalisation is completed by arranging that the non-zero elements, if any, in each block $B_{u,v}$ (or $\bar{B}_{n,s}$) occur in the first rows and columns of $B_{u,v}$ (or $\bar{B}_{n,s}$) which do not contain non-zero elements in any block $B_{u',v'}$ such that $u' < u$, $v' \leqslant v$ or $u' \leqslant u$, $v' < v$ (or $B_{n',s'}$ such that $n' < n$, $s' \leqslant s$ or $n' \leqslant n$, $s' < s$).

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REFERENCES

[1] S. C. Chang, *Homotopy invariants and continuous mappings*, Proc. Roy. Soc., (British), **A 202** (1950), 253-263.

[2] S. C. Chang and J. H. C. Whitehead, *Note on cohomology systems*, Quart. J. Math., Oxford, (2), **2** (1951), 167-174.

[3] J. H. C. Whitehead, *On simply connected 4-dimensional polyhedra*, Comm. Math. Helv., **22** (1949), 48-92.

[4] J. Adem, *The iteration of the Steenrod squares in algebraic topology*, Proc. N. A. S., **38** (1952), 720-726.

*) Of course, the elements of the matrix are all 0 or 1.

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MATHEMATICS

Equation $(Vl)^2=1$ in the Complex Domain

by

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From the point of view of possible applications in optics, it seems interesting to investigate the equation

$$(Vl)^2=1, \tag{1}$$

where l is a complex function of real variables.

We shall consider the case of two variables. For the most part the case of three variables leads to similar considerations.

Equation (1) is equivalent to the system of two real equations (2) and (3), obtained by the substitution of $l=u+iv$ in equation (1):

$$(Vu)^2-(Vv)^2=1, \tag{2}$$

$$Vu\cdot Vv=0. \tag{3}$$

It does not seem easy to make an analysis in the real domain, i. e. for the system of equations (2) and (3). The analysis given below makes use of the complex form (1) of the equation in question.

We extend the concept of complete integral to the complex domain. For equation (1) it will have the form

$$l=ax+by+c, \tag{4}$$

where a, b, c are complex numbers satisfying the relation

$$a^2+b^2=1. \tag{5}$$

Let us consider the Cauchy problem:

$$x=x_0(r), \tag{6}$$

$$y=y_0(r), \tag{7}$$

$$l=l_0(r)=u_0(r)+iv_0(r), \tag{8}$$

$$\dot{x}_0^2+\dot{y}_0^2=1, \tag{9}$$

$$r\varepsilon\langle\alpha,\beta\rangle. \tag{10}$$

We assume that the real functions of a real variable x_0, y_0, u_0, v_0 are, with respect to the complex variable $t=r+is$, regular in a certain open set containing a segment of the real axis $\langle \alpha, \beta \rangle$.

Now we determine three analytical functions, a, b, c, of the variable t from the system of equations

$$u_0 + iv_0 = ax_0 + by_0 + c, \tag{11}$$

$$u_0' + iv_0' = ax_0' + by_0', \tag{12}$$

$$a^2 + b^2 = 1. \tag{13}$$

The calculations can be carried out. We obtain:

$$a = l_0' x_0' + y_0' \sqrt{1-(l_0')^2}, \tag{14}$$

$$b = l_0' y_0' - x_0' \sqrt{1-(l_0')^2}, \tag{15}$$

$$c = l_0 - ax_0 - by_0. \tag{16}$$

In addition we assume that for real $t \varepsilon \langle \alpha, \beta \rangle$

$$(l_0')^2 \neq 1. \tag{17}$$

The functions thus obtained will be regular in a certain neighbourhood of the segment $\langle \alpha, \beta \rangle$ of the real axis.

We denote by $\bar{\tau}$ the unit vector tangent to the curve x_0, y_0; then, for the solution of the Cauchy problem, l

$$l_0' = \bar{\tau} \nabla l. \tag{18}$$

Thus, condition (17) will exclude the case $\nabla v = 0$, ∇u tangent to the curve x_0, y_0. This condition corresponds in the real domain to the elimination of the characteristic direction of the initial curve.

We shall obtain the solution of the Cauchy problem by eliminating the variable t from the system of equations

$$l = ax + by + c, \tag{19}$$

$$a'x + b'y + c' = 0. \tag{20}$$

It is, however, interesting that in the parametric form — as we shall see presently under fairly general assumptions — the equation of integral surface can be determined effectively, which is not often the case with partial differential equations.

We may regard equation (20) as a system of two real equations and determine x and y from this system. We introduce the following notations for the real and the imaginary parts of the functions a', b' and c':

$$a' = u_1^{(1)} + iv_1^{(1)}; \quad b' = u_2^{(1)} + iv_2^{(1)}; \quad c' = u_2^{(1)} + iv_3^{(1)}. \tag{21}$$

Since $t=r+is$, we can represent this system in the form

$$u_1^{(1)}(r,s)x+u_2^{(1)}(r,s)y+u_3^{(1)}(r,s)=0, \tag{22}$$

$$v_1^{(1)}(r,s)x=v_2^{(1)}(r,s)y+v_3^{(1)}(r,s)=0. \tag{23}$$

The determinant of this system is different from 0, provided that

1^0 one of the functions a' and b' is different from zero and

2^0 one of the functions a and b is complex.

For instance let $a'\neq 0$. Then, if the determinant of the system equals zero, the ratio b'/a' is real. Differentiating equation (13) we obtain the relation

$$\frac{b'}{a'}=-\frac{a}{b}. \tag{24}$$

Hence the ratio a/b is also real. Finally it follows from (13) that a and b are real.

Assuming that conditions 1^0 and 2^0 are satisfied on the initial curve and determining x and y as functions of the parameters r and s, we obtain a parametric equation of the integral surface (19).

Moreover, it must be shown that equation (20), or the equivalent system of equations (22) and (23), determines t as a function of x and y.

It follows from equations (11) and (12) that for each point of the initial curve the system of equations (22), (23) is satisfied by $t=r$ and $x=x_0(r)$, $y=y_0(r)$.

$$u_1^{(1)}(r,0)x_0(r)+u_2^{(1)}(r,0)y_0(r)+u_3^{(1)}(r,0)=0, \tag{25}$$

$$v_1^{(1)}(r,0)x_0(r)+v_2^{(1)}(r,0)y_0(r)+v_3^{(1)}(r,0)=0. \tag{26}$$

It remains to investigate the Jacobian formed by the partial derivatives with respect to r and s of the left-hand sides of equations (22), (23). Taking into account the Cauchy-Riemann equations and the relations obtained by differentiating equations (25), (26) with respect to r, we find that, for the argument values that interest us, this Jacobian is

$$|a'x_0'+b'y_0'|^2. \tag{27}$$

We shall prove that if condition 1^0 is satisfied, this Jacobian is different from 0. If expression (27) were equal to 0, we should have $a'x_0'+b'y_0'=0$. Then, using equation (13) it could be shown that the vector $[a,b]$ is a real vector satisfying the relation $aa'+bb'=0$. Since $|a'|^2+|b'|^2>0$, it follows that the vectors $[x_0', y_0']$ and $[a,b]$ are parallel. Consequently, in view of $x_0'^2+y_0'^2=a^2+b^2=1$, we would obtain $(ax_0'+by_0')^2=1$, i. e. $(l_0')^2=1$, which contradicts assumption (17).

When condition 1^0 is satisfied but condition 2^0 is not, the solution may be obtained by eliminating the parameter t. It can easily be seen

that t is defined as a function of x and y under the assumptions made above not only in the neighbourhood of each point of the initial curve but also in a certain open set containing the whole curve. This conclusion can be drawn by a similar reasoning to that applied in the proof of the uniqueness of monodrome continuation of an analytic function.

The assumption that condition 1^0 is satisfied is not necessary for the Cauchy problem to be solvable, as can be seen by the example

$$a' \equiv b' \equiv 0. \tag{28}$$

In that case the elimination of the parameter is not needed. The assumption under discussion made it possible to simplify the proof. Substituting the t calculated from equation (20) in function (19), we obtain a function which satisfies equation (1) and the initial conditions.

The lines of fields ∇u and ∇v play an important role with regard to the applications of the differential equation here considered. These curves show a certain analogy to the projections of characteristics upon the plane xy in the case of an equation in the real domain. We shall outline the procedure leading to the analytical definition of these curves.

We adopt the following symbols for the coordinates of the vectors: $\nabla u = [u_1, u_2]$, $\nabla v = [v_1, v_2]$. The following relations hold:

$$u_1 = \operatorname{Re} a, \quad u_2 = \operatorname{Re} b, \quad v_1 = \operatorname{Im} a, \quad v_2 = \operatorname{Im} b. \tag{29}$$

Since, according to our previous considerations, a and b may be regarded as functions of x and y, the differential equations defining the lines of the field ∇u can be expressed in the form

$$\frac{dy}{dx} = \frac{u_2(x, y)}{u_1(x, y)}, \tag{30}$$

and an analogous equation can be obtained for the lines of the field ∇v.

Still, a and b, and consequently u_1, u_2, v_1, v_2, cannot as a rule be expressed explicitly as functions of x and y. The corresponding differential equations can be obtained in a distinct form when it is possible to make use of the previously introduced parameters r and s. The problem of finding the curves in question can be reduced of course to determining a suitable relation between r and s, say in the form of the function $r = f(s)$. Substituting $f(s)$ for r in the solutions $x(r, s)$, $y(r, s)$ of the system of equations (22), (23), we obtain a parametric equation of the curve.

The differential equation defining the function f may be obtained, for example, in the following way. Owing to the relation obtained by differentiating (13) $au' + bb' = 0$, equation (20) multiplied by $-a/b'$ gives the relation

$$bx - ay - \frac{c'a}{b'} = 0. \tag{31}$$

We denote the real part and the imaginary part $-c'a/b'$ by u_3 and v_3 respectively. Equation (31) is then equivalent to the system of real equations

$$u_2x-u_1y+u_3=0, \tag{32}$$

$$v_2x-v_1y+v_3=0. \tag{33}$$

These equations are identically satisfied by $x(r,s)$ and $y(r,s)$. Let $r=f(s)$ be the equation of a line of the field Vu. We substitute $r=f(s)$ in identity (32) and differentiate with respect to s. Owing to the assumption made with regard to the function f, some of the components will drop off (as a result of a relation corresponding to (30) but having different arguments), and we shall obtain

$$(u_{2r}f'+u_{2s})x-(u_{1r}f'+u_{1s})y+u_{3r}f'+u_{3s}=0. \tag{34}$$

In virtue of equations (21) and (29), the expression obtained may be given a different form, namely

$$(u_2^{(1)}f'-v_2^{(1)})x-(u_1^{(1)}f'-v_1^{(1)})y+u_{3r}f'+u_{3s}=0. \tag{35}$$

Finally, in view of (22), (23) and (35), we obtain the differential equation of function f

$$\begin{vmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} \\ v_1^{(1)} & v_2^{(1)} & v_3^{(1)} \\ (u_2^{(1)}f'-v_2^{(1)}) & (-u_1^{(1)}f'+v_1^{(1)}) & (u_{3r}f'+u_{3s}) \end{vmatrix}=0. \tag{36}$$

Choosing a suitable value of $r=f(0)$ we obtain the line of the field passing by the required point of the initial curve.

Making use of equation (33) we can establish in a similar way the differential equation of the family of lines of the field Vv. Instead of using equation (31) we could solve the system of equations (22), (23). Naturally it remains an open question which of the methods here presented will be more effective in a concrete case.

In conclusion, I should like to thank J. Plebański for his remarks, which have been very valuable for the analysis of the problems here considered.

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MATHEMATICS

On the Estimation of the Existence Domain for Solutions of a Non-Linear Partial Differential Equation of the First Order

by

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In the present note we give a method of determining "along a characteristic" the region where a solution exists for equation

$$\frac{\partial z}{\partial x}=f\left(x,y_1,\ldots,y_n,z,\frac{\partial z}{\partial y_1},\ldots,\frac{\partial z}{\partial y_n}\right) \tag{1}$$

with the initial condition

$$z(a,y_1,\ldots,y_n)=\omega(y_1,\ldots,y_n) \tag{2}$$

(Theorem 1).

Moreover, as an application of Theorem 1, we shall estimate the region where solutions exist for equation

$$\frac{\partial z}{\partial x}=\tilde{f}\left(x,y_1,\ldots,y_n,\frac{\partial z}{\partial y_1},\ldots,\frac{\partial z}{\partial y_n}\right) \tag{3}$$

with initial condition (2) (Theorem 2).

Notation. We shall denote by $Y(x,V)=(y_1(x,V),\ldots,y_n(x,V))$, $s(x,V)$, $Q(x,V)=(q_1(x,V),\ldots q_n(x,V))$, $t_{ij}(x,V)$ $(i,j=1,\ldots,n)(V=(v_1,\ldots,v_n))$ the solution of the system U of ordinary differential equations consisting of the system

$$\begin{cases} y_i'=-f_{q_i}(x,Y,s,Q) & (i=1,\ldots,n) \\ s'=f(x,Y,s,Q)-\sum_{k=1}^{n} q_k f_{q_k}(x,Y,s,Q), \\ q_i'=f_{y_i}(x,Y,s,Q)+q_i f_z(x,Y,s,Q) & (i=1,\ldots,n), \end{cases} \tag{4}$$

and the system

$$(5)\qquad t'_{ij}=\sum_{k=1}^{n}\sum_{m=1}^{n} f_{q_kq_m}(x,Y,s,Q)t_{ki}\,t_{mj}+\sum_{k=1}^{n}(f_{q_ky_i}(\ldots)+f_{q_kz}(\ldots)q_i)t_{kj}+$$

$$+\sum_{k=1}^{n}(f_{q_ky_j}(\ldots)+f_{q_kz}(\ldots)q_j)t_{ki}+f_z(\ldots)t_{ij}+$$

$$+f_{y_iy_j}(\ldots)+f_{y_iz}(\ldots)q_j+f_{y_jz}(\ldots)q_i+f_{zz}(\ldots)q_iq_j\qquad (i,j=1,\ldots,n)\ *),$$

satisfying the initial conditions

$$(6)\qquad \begin{aligned} &Y(a,V)=V,\ s(a,V)=\omega(V),\ q_i(a,V)=\omega_{y_i}(V)\qquad (i=1,\ldots,n)\\ &t_{ij}(a,V)=\omega_{y_iy_j}(V)\qquad (i,j=1,\ldots,n).\end{aligned}$$

THEOREM 1. *Let the function $f(x,Y,z,Q)$ be defined and of class C^p $(p\geqslant 2)$ in a $(2n+2)$-dimensional open set, and let the function $\omega(Y)$ be defined and of class C^p in a certain n-dimensional open set. Let the solution $Y(x,B)$, $s(x,B)$, $Q(x,B)$, $t_{ij}(x,B)$ $(i,j=1,\ldots,n)$, where B is a fixed point, be defined in the interval, $a_1<x<a_2$ $(a_1<a<a_2)$ **).*

Then there exists a solution $z(x,Y)$ of equation (1) of class C^p in a certain open set containing the arc $Y=Y(x,B)$ $a_1<x<a_2$, satisfying initial condition (2) in a certain neighbourhood of the point $Y=B$.

Moreover, the following identities are satisfied:

$$(7)\qquad z_{y_iy_j}(x,Y(x,B))=t_{ij}(x,B)\qquad (i,j=1,\ldots,n)\quad a_1<x<a_2;$$

the functions $u_i^{(j)}(x,V)=\dfrac{\partial y_i(x,V)}{\partial v_j}$ $(i=1,\ldots,n)$ $1\leqslant j\leqslant n$ satisfy the system of ordinary differential equations

$$(8)\quad u'_i=\sum_{k=1}^{n}\Big(-f_{q_iy_k}(W)-f_{q_iz}(W)q_k(x,V)-\sum_{m=1}^{n}f_{q_iq_m}(W)t_{mk}(x,V)\Big)u_k,\quad (i=1,\ldots,n),$$

where

$$W=(x,Y(x,V),\ s(x,V),\ Q(x,V)),$$

for $a_1<x<a_2$ $V=B$, and

$$(9)\qquad \det\big(u_i^{(j)}(x,B)\big)\neq 0\qquad \textit{for}\qquad a_1<x<a_2.$$

We shall give the main idea of the proof for $p\geqslant 3$ ***).

*) Analogous systems have been used for the quasilinear partial differential equation of the first order in papers [1], [2].

**) It follows that the function $\omega(Y)$ is defined for $Y=B$, and the function $f(x,Y,z,Q)$ is defined at the points $x=\xi$, $y=Y(\xi,B)$, $z=s(\xi,B)$, $Q=Q(\xi,B)$ for $a_1<\xi<a_2$.

***) In case $p=2$ we can obtain Theorem 1 by a suitable approximation of the function f, ω by functions of class C^3.

For the function $t_{ij}^*(x)=z_{y_iy_j}(x,Y(x,B))$, defined on an interval $A=(a',a'')(a_1\leqslant a'<a<a''\leqslant a_2)$, where $z(x,Y)$ is a function defined and of class C^p on an open set and satisfying equation (1) and condition (2), we have, in view of (4),

$$\frac{d}{dx}t_{ij}^*(x)=z_{y_iy_jx}\big(x,Y(x,B)\big)-\sum_{k=1}^{n}z_{y_iy_jy_k}f_{q_k}\big(x,Y(x,B),\ s(x,B),\ Q(x,B)\big).$$

(By classical theory there exists a solution of class C^p of equation (1) satisfying condition (2) and defined in a certain neighbourhood of the point $x=a,\ Y=B$).

In view of the identity

$$z_{y_iy_jx}=\frac{\partial^2}{\partial y_i\partial y_j}\big(f\big(x,Y,z(x,Y),\ z_{y_1}(x,Y),\dots,z_{y_n}(x,Y)\big)\big),$$

it is easy to ascertain that the functions $t_{ij}=t_{ij}^*(x)$ satisfy system (5), whence we can easily deduce that the identities (7) hold in A.

Differentiating the identity

$$y_i'(x,V)=-f_{q_i}\big(x,Y(x,V),\ z\big(x,Y(x,V)\big),\ z_{y_1}\big(x,Y(x,V)\big),\dots,z_{y_n}\big(x,Y(x,V)\big)\big)$$

on both sides with respect to v_j, and making use of identities $z(x,Y(x,V))=s(x,V)$, $z_{y_i}(x,Y(x,V))=q_i(x,V)$ $(i=1,\dots,n)$ and identities (7), we obtain (in A) equations (8).

Relation (9) follows from relation (8) and the identity $Y(a,V)=V$.

We can prove that the properties stated in Theorem 1 hold in the whole interval $a_1<x<a_2$ by showing that its largest subinterval containing the point a, in which those properties hold, is both open and closed in that interval.

Remark 1. The assumption that the function $f(x,Y,z,Q)$ is of class C^p was made because of its simple wording; it may be replaced by a weaker assumption

Theorem 2. *Suppose that for* $|x-a|<\gamma$ $(\gamma>0)$ *the function* $\bar{f}(x,r_1,\dots,r_{2n})$, *together with its derivatives of the first order with respect to the variables* $r_1,\dots,r_{2n}$, *is continuous and bounded and satisfies conditions*

$$|\bar{f}_{r_i}(x,r_1,\dots,r_{2n})-\bar{f}_{r_i}(x,\tilde{r}_1,\dots,\tilde{r}_{2n})|\leqslant K\sum_{j=1}^{2n}|r_j-\tilde{r}_j|, \tag{10}$$

for any $r_1,\dots,r_{2n}$, $\tilde{r}_1,\dots,\tilde{r}_{2n}$ $(i=1,\dots,2n)$, *where* K *is a certain positive constant, and the function* $\omega(y_1,\dots,y_n)$ *together with its derivatives of the first order is continuous and bounded in the space* E_n, *and satisfies conditions*

$$|\omega_{y_i}(y_1,\dots,y_n)-\omega_{y_i}(\tilde{y}_1,\dots,\tilde{y}_n)|\leqslant J\sum_{k=1}^{n}|y_k-\tilde{y}_k| \tag{11}$$

for any $y_1,\dots,y_n$, $\tilde{y}_1,\dots,\tilde{y}_n$ $(i=1,\dots,n)$.

Under the above assumptions there exists a solution $z(x,y_1,\dots,y_n)$ *of equation* (3), *with initial condition* (2), *which is of class* C^1 *for* $|x-a|<\delta$ *and satisfies the conditions*

$$|z_{y_i}(x,y_1,\dots,y_n)-z_{y_i}(x,\bar{y}_1,\dots,\bar{y}_n)|\leqslant T(x)\sum_{j=1}^{n}|y_j-\bar{y}_j| \quad (i=1,\dots,n) \tag{12}$$

for any $y_1,\dots,y_n,\bar{y}_1,\dots,\bar{y}_n$, *and* $|x-a|<\delta$,

where $T(x)=\dfrac{J+K(1+nJ)|x-a|}{1-nK(1+nJ)|x-a|}$ *and* $\delta=\min\left(\gamma,\dfrac{1}{nK(1+nJ)}\right)$.

Remark 2. In Theorem 2 the constant δ cannot be replaced by a larger constant. For simplicity, in stating Theorem 2 we have confined ourselves to equation (3), which is a particular case of equation (1). In paper [3] is included a different estimation (by a different method) of the existence region of the solution of equation (1) under analogous assumptions of regularity to those used in Theorem 2.

In the proof of Theorem 2 we can make the additional assumption that the functions $\bar{f}$, ω are of class C^2 (or C^3). This assumption may be eliminated by a suitable approximation of the functions $\bar{f}$, ω by the functions $\tilde{f}$, $\tilde{\omega}$ of class C^2 (or C^3), satisfying the inequalities

$$|\tilde{\omega}_{y_iy_j}|\leqslant J \quad (i,j=1,\dots,n), \tag{13}$$

$$|\tilde{f}_{x_ix_j}|\leqslant K \quad (i,j=1,\dots,2n) \tag{14}$$

(cf. (11), (10)). Then it is enough to prove Theorem 2 for $\bar{f}=\tilde{f}$, $\omega=\tilde{\omega}$.

In order to apply Theorem 1 to the proof of Theorem 2 we must ascertain that for any V the functions $Y(x,V)$, $s(x,V)$, $Q(x,V)$, $t_{ij}(x,V)$, corresponding to $f=\tilde{f}$, $\omega=\tilde{\omega}$, are defined in the interval $|x-a|<\gamma$. It can easily be observed that the functions $Y(x,V)$, $s(x,V)$, $Q(x,V)$ are defined in the whole interval $|x-a|<\gamma$. Making use of the fact that the functions $t_{ij}(x,V)$, $(i,j=1,\dots,n)$ satisfy system (5) (which is considerably simplified for equation (3)) and initial conditions (6), and taking into account relations (13), (14), we can easily ascertain that for any V the function $M(x,V)=\max\limits_{ij}|t_{ij}(x,V)|$ satisfies the differential inequality $|D^+M\leqslant K(n^2M^2+2nM+1)$, where D^+M denotes the right-hand derivative of function M with respect to x, and the inequality $|M(a;V)|\leqslant J$. Note further that for $x>a$ the function $T(x)$ satisfies the differential equation $T'=K(n^2T^2+2nT+1)$, for $x<a$ it satisfies the equation $T'=-K(n^2T^2+2nT+1)$, and for $x=a$ the condition $T(a)=J$. Applying to the integrals $t_{ij}(x,V)$ the theorem on integrals extending to the boundary and to the functions $M(x;V)$, $T(x)$ the theorem on differential inequality, we can thus ascertain that the functions $t_{ij}(x,V)$ are defined for $|x-a|<\delta$ and satisfy the inequalities

$$t_{ij}(x,V)\leqslant T(x) \quad (i,j=1,\dots,n). \tag{15}$$

Applying Theorem 1 it is not difficult to show that for $|x-a|<\delta$,

$$\operatorname{Det}\left(\frac{\partial y_i(x,V)}{\partial v_j}\right)>\beta(x)>0,$$

$$\sum_{ij}\left(\frac{\partial y_i(x,V)}{\partial v_j}\right)^2\leqslant\Gamma(x)$$

for any V, where $\beta(x), \Gamma(x)$ are functions independent of V. Accordingly, the transformation $Y=Y(x,V)$ has an inverse transformation $V=V(x,Y)$, defined in the whole space of points Y. (Cf. [4] Theorem 3 for $R=+\infty$ taking into account note 24). It is easy to observe that the function $z(x,Y)=s(x,V(x,Y))$ is the required solution. Condition (12) follows from identities (7) and inequalities (15).

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REFERENCES

[1] J. Perausówna, *Sur le domaine d'existence des intégrales de l'équation* $p+f(x,y,z)q=g(x,y,z)$, Ann. Soc. Polon. Math. **12** (1933), p. 1.

[2] T. Ważewski, *Sur le domaine d'existence des intégrales de l'équation aux dérivées partielles du premier ordre linéaire*, ibidem p. 6.

[3] — *Über die Bedingungen der Existenz der Integrale partieller Differentialgleichungen erster Ordnung*, Math. Zeit. **43** (1938), 322.

[4] — *Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes*, Ann. Soc. Polon. Math. **20** (1947), 81.

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MATHÉMATIQUE

Sur la méthode de A. Pliś de déterminer le domaine d'existence de la solution du problème de Cauchy pour les équations aux dérivées partielles du premier ordre

par

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1. Considérons l'équation

$$\frac{\partial z}{\partial x}=f\left(x,y_1,\dots,y_n,z,\frac{\partial z}{\partial y_1},\dots,\frac{\partial z}{\partial y_n}\right) \tag{1.1}$$

On peut résoudre, par la méthode des caractéristiques, le problème de Cauchy, c'est-à-dire celui de trouver la solution

$$z(x,y_1,\dots,y_n) \tag{1.2}$$

de l'équation (1.1) satisfaisant à la condition initiale

$$z(a,y_1,\dots,y_n)=\omega(y_1,\dots,y_n), \tag{1.3}$$

où ω est une fonction donnée.

On cherche à cet effet la solution (4.1) du système de caractéristiques (3.1) assujettie aux conditions initiales (4.3). Pour établir le plus grand domaine dans lequel la solution (1.2), satisfaisant à (1.3), existe, il convient de déterminer un intervalle $\Delta(V)$ aussi grand que possible et dans lequel l'inégalité (4.7) est encore vraie. A. Pliś a résolu ce problème en introduisant les équations différentielles accessoires (3.2) qui, jointes aux équations de caractéristiques du premier ordre (3.1), forment le système U embrassant les deux systèmes (3.1) et (3.2). Les solutions du système U constituent les caractéristiques du second ordre. Soit $\Delta(V)$ l'intervalle maximum dans lequel existe la caractéristique du second ordre (4.1), (4.2) du système U assujettie aux conditions initiales (4.3), (4.4). Le théorème de A. Pliś [1] affirme que l'inégalité (4.7) a lieu dans l'intervalle $\Delta(V)$ sous condition que f et ω satisfont aux hypothèses H et K (voir plus loin, **2**).

Ce résultat fondamental et, en un certain sens, définitif, fournit une nouvelle méthode pour déterminer le domaine d'existence de la solution du problème de Cauchy. Parmi les méthodes connues, celle de A. Pliś est la meilleure.

Brève et élègante, la démonstration de A. Pliś n'est cependant pas directe. Elle présuppose la connaissance des théorèmes classiques sur l'existence et sur la régularité de la solution du problème de Cauchy. Elle s'appuie en outre sur certains théorèmes concernant la façon dont varie la solution du problème de Cauchy lorsque l'on fait varier la fonction ω intervenant dans la condition initiale (1.3).

La communication présente apporte une démonstration directe du théorème, en se servant d'un artifice analogue à un artifice classique de Cauchy et sans faire intervenir l'équation (1.1), mais opérant exclusivement sur les équations U des caractéristiques du second ordre. On peut établir par cette voie l'existence de la solution du problème de Cauchy et, en même temps, déterminer le domaine d'existence de cette solution.

2. Posons

$$Y=(y_1,\dots,y_n), \qquad Q=(q_1,\dots,q_n) \qquad \text{et} \qquad V=(v_1,\dots,v_n).$$

Hypothèse H. *La fonction de $2n+2$ variables réelles*

$$f(x,y_1,\dots,y_n,z,q_1,\dots,q_n),$$

c'est-à-dire la fonction $f(x,Y,z,Q)$, est continue dans un ensemble ouvert B et ses dérivées partielles du second ordre relatives aux $2n+1$ variables y_i,z,q_i l'y sont également.

Hypothèse K. *La fonction*

$$\omega(v_1,\dots,v_n),$$

c'est-à-dire la fonction $\omega(V)$, est de classe C^2 dans un ensemble ouvert A et le point

$$x=a, \qquad Y=V, \qquad z=\omega(V), \qquad q_i=\frac{\partial\omega(v_1,\dots,v_n)}{\partial v_i}$$

(où $i=1,\dots,n$), appartient à B lorsque V appartient à A.

3. Envisageons deux systèmes suivants des équations différentielles ordinaires:

$$\frac{dy_i}{dx}=-f_{q_i}(x,Y,z,Q) \qquad \text{pour} \qquad i=1,\dots,n,$$

$$\frac{dz}{dx}=f(x,Y,z,Q)-\sum_{k=1}^{n}q_k f_{q_k}(x,Y,z,Q), \tag{3.1}$$

$$\frac{dq_i}{dx}=f_{y_i}(x,Y,z,Q)+q_i f_z(x,Y,z,Q) \qquad \text{pour} \qquad i=1,\dots,n$$

et

$$\frac{dt_{ij}}{dx}=f_{y_iy_j}+f_{y_iz}q_j+f_{y_jz}q_i+f_{zz}q_iq_j+f_zt_{ij}+ \tag{3.2}$$

$$+\sum_{k=1}^{n}(f_{y_iq_k}t_{kj}+f_{y_jq_k}t_{ki})+\sum_{k=1}^{n}f_{zq_k}(q_it_{kj}+q_jt_{ki})+$$

$$+\sum_{k=1}^{n}\sum_{s=1}^{n}f_{q_kq_s}t_{ki}t_{sj}\qquad(i,j=1,\dots,n),$$

où $f_{y_iy_j}=\partial^2 f/\partial y_i\partial y_j$, etc. Le système composé de systèmes (3.1) et (3.2) sera désigné par U. Dans le système (3.1), il y a $2n+1$ fonctions inconnues, à savoir y_i, z et q_i, et dans le système U interviennent en outre n^2 fonctions t_{ij} où $i,j=1,\dots,n$. Le système U contient donc n^2+2n+1 fonctions inconnues.

4. Théorème (de A. Pliś). *Admettons sur $f(x,Y,z,Q)$ les hypothèses H et K. Désignons par*

$$Y(x,V)=\big(y_1(x,V),\dots,y_n(x,V)\big),\ z(x,V),\ Q(x,V)=\big(q_1(x,V),\dots,q_n(x,V)\big) \tag{4.1}$$

et

$$t_{ij}(x,V),\qquad \text{où}\qquad i,j=1,\dots,n, \tag{4.2}$$

la solution du système U satisfaisant aux conditions initiales

$$Y(a,V)=V,\ z(a,V)=\omega(V),\qquad q_i(a,V)=\omega_{v_i}(V),\quad \text{où}\quad l=i,\dots,n, \tag{4.3}$$

$$t_{ij}(a,V)=\omega_{v_iv_j}(V),\qquad \text{où}\qquad i,j=1,\dots,n \tag{4.4}$$

et par $\Delta(V)$ l'intervalle maximum (ouvert) dans lequel la solution (4.3), (4.4) *du système U existe. On a $a\in\Delta(V)$.*

Envisageons le système de n équations linéaires homogènes

$$\frac{du_i}{dx}=\sum_{k=1}^{n}\Big\{-f_{q_iy_k}(W)-f_{q_iz}(W)q_k(x,V)-\sum_{m=1}^{n}f_{q_iq_m}(W)t_{mk}(x,V)\Big\}u_k, \tag{4.5}$$

où

$$W=\big(x,Y(x,V),\ z(x,V),\ Q(x,V)\big)\qquad\text{et}\qquad i=1,\dots,n.$$

Nous affirmons que les propriétés suivantes P_1 et P_2 ont lieu:

Propriété P_1. Pour chaque $j=1,\dots,n$ fixe, le système d'équations (4.5) se trouve satisfait dans l'intervalle $\Delta(V)$ en posant

$$u_i=\frac{\partial y_i(x,V)}{\partial v_j}\qquad\text{pour}\qquad i=1,\dots,n. \tag{4.6}$$

Propriété P_2. Le jacobien en $v_1,\dots,v_n$ de la suite de fonctions $y_1(x,V),\dots,y_n(x,V)$ n'est pas nul pour $x\in\Delta(V)$, c'est-à-dire

$$\det\left(\frac{\partial y_i(x,V)}{\partial v_j}\right)\neq 0. \tag{4.7}$$

Idée de la démonstration. La propriété P_2 est une conséquence de la propriété P_1. Admettons en effet P_1. Les formules (4.6) fournissent (pour $j=1,\dots,n$) n solutions différentes (4.5), définies dans l'intervalle $\Delta(V)$ du système de n équations linéaires et homogènes. Pour $x=a$, le déterminant (4.7) est non nul (il est égal à 1). Comme $a \in \Delta(V)$, il en résulte, suivant un théorème bien connu, que l'inégalité (4.7) a lieu dans $\Delta(V)$.

Il suffit donc d'établir la propriété P_1. Elle résulte par une voie directe et élémentaire du lemme qui suit.

5. LEMME. *Les hypothèses et les notations du théorème étant admises, on a les identités suivantes dans l'intervalle* $\Delta(V)$:

$$t_{ij}(x,V)=t_{ji}(x,V) \quad \textit{pour} \quad i,j=1,\dots,n. \tag{5.1}$$

$$\frac{\partial z(x,V)}{\partial v_k}=\sum_{j=1}^{n} q_j(x,V)\frac{\partial y_j(x,V)}{\partial v_k} \quad \textit{pour} \quad k=1,\dots,n, \tag{5.2}$$

$$\frac{\partial q_i(x,V)}{\partial v_k}=\sum_{j=1}^{n} t_{ij}(x,V)\frac{\partial y_j(x,V)}{\partial v_k} \quad \textit{pour} \quad i,k=1,\dots,n. \tag{5.3}$$

Démonstration. Introduisons les fonctions $r_{ij}(x,V)$, $L_k(x,V)$ et $M_{ik}(x,V)$ de façon à avoir dans $\Delta(V)$ les identités

$$r_{ij}(x,V)=t_{ij}(x,V)-t_{ji}(x,V),$$

$$\frac{\partial z(x,V)}{\partial v_k}=\sum_j q_j(x,V)\frac{\partial y_j(x,V)}{\partial v_k}+L_k(x,V), \tag{5.4}$$

$$\frac{\partial q_i(x,V)}{\partial v_k}=\sum_j t_{ij}(x,V)\frac{\partial y_j(x,V)}{\partial v_k}+M_{ik}(x,V). \tag{5.5}$$

Les fonctions (4.1) et (4.2) satisfont aux équations (3.1) et (3.2). Il en résulte pour $x \in \Delta(V)$ que

$$\frac{dr_{ij}(x,V)}{dx}=f_z(W)r_{ij}(x,V) \tag{5.6}$$

et, en vertu de (4.4), que $r_{ij}(a,V)=\omega_{v_iv_j}(V)-\omega_{v_jv_i}(V)=0$. Les équations (5.6) étant linéaires et homogènes, il en résulte que $r_{ij}(x,V)=0$ pour $x \in \Delta(V)$, ce qui entraîne les identités (5.1).

Pour établir les identités (5.2) et (5.3) substituons les fonctions (4.1) et (4.2) dans les équations (3.1) et (3.2) et différencions (3.1) relativement à v_k. Il vient ainsi pour $x \in \Delta(V)$ les identités suivantes:

$$\frac{d}{dx}\left(\frac{\partial y_i(x,V)}{\partial v_k}\right)=\frac{\partial}{\partial v_k}\{f_{q_i}(W)\} \quad \text{où} \quad i=1,\dots,n, \tag{5.7}$$

$$(5.8)\qquad \frac{d}{dx}\left(\frac{\partial z(x,V)}{\partial v_k}\right)=\frac{\partial}{\partial v_k}\left\{f(W)-\sum_{j=1}^{n} q_j(x,V)f_{q_j}(W)\right\},$$

$$(5.9)\qquad \frac{d}{dx}\left(\frac{\partial q_i(x,V)}{\partial v_k}\right)=\frac{\partial}{\partial v_k}\{f_{y_i}(W)+q_i(x,V)f_z(W)\} \quad \text{où} \quad i=1,\dots,n.$$

En effectuant les différenciations des membres droits de ces identités, en y substituant ensuite (5.4) et (5.5) et en tenant compte des identités (5.1) et (3.2), on obtient pour $x\in\Delta(V)$ les identités

$$(5.10)\qquad \frac{dL_k(x,V)}{dx}=f_z(W)L_k(x,V) \quad \text{pour} \quad k=1,\dots,n.$$

$$(5.11)\qquad \frac{d}{dx}M_{ik}(x,V)=\left\{q_i f_{zz}+f_{y_i z}+\sum_j t_{ij}f_{q_j z}\right\}^{*}L_k$$

$$+\{f_z\}^{*}M_{ik}+\sum_j\left\{f_{y_i q_j}+q_i f_{z q_j}+\sum_s f_{q_s q_j}t_{is}\right\}^{*}M_{jk} \quad \text{pour} \quad i,k=1,\dots,n,$$

où l'astérisque désigne la substitution des fonctions (4.1) et (4.2) dans l'expression entre { }. Or, en raison de (4.3), on a

$$L_k(a,V)=0 \quad \text{et} \quad M_{ik}(a,V)=0.$$

Le système (5.10), (5.11) étant linéaire et homogène, il en résulte que l'on a $L_k(x,V)=0$ et $M_{ik}(x,V)=0$ pour $x\in\Delta(V)$.

Les identités (5.4) et (5.5) conduisent donc aux identités (5.2) et (5.3).

6. Démonstration du théorème. En différenciant le membre droit de (5.7) et en y substituant les expressions (5.2) et (5.3), on obtient (pour $x\in\Delta(V)$) une identité exprimant que les fonctions (4.6) satisfont (pour $x\in\Delta(V)$) au système (4.5). La démonstration de la propriété P_1 et, par conséquent, du théorème est ainsi achevée.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] A. Pliś, *On the estimation of the existence domain for solutions of a nonlinear partial differential equation of the first order*, travail précédent dans le présent numéro du Bulletin.

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MATHÉMATIQUE

Équations différentielles à coefficients constants considérées dans les espaces linéaires généraux

par

J. MIKUSIŃSKI

Présenté par T. WAŻEWSKI le 12 Décembre 1955

1. Dans certaines théories, par exemple dans le calcul opérationnel [2], [3], on considère des équations différentielles linéaires dont les coefficients sont des éléments d'un corps plus général que celui des nombres. Il existe plusieurs démonstrations [1], [4], [6] que, sous certaines hypothèses bien naturelles, l'équation d'ordre n a au plus n solutions linéairement indépendantes. On sait qu'en général le nombre de ces solutions peut être inférieur à n. On ne connaît cependant pas de théorèmes généraux sur le nombre exact de ces solutions; ce problème est résolu dans cette note *). Les démonstrations seront publiées dans Studia Mathematica.

2. Nos considérations concernent un espace linéaire $\mathcal{E}$ quelconque. On suppose seulement que les coefficients constituent un corps commutatif $\mathcal{C}$ de caractéristique 0.

On suppose qu'une opération de dérivation est définie pour certains éléments de $\mathcal{E}$ et que cette opération jouit des propriétés suivantes:

1° Si x' et y' existent, on a $(ax+by)'=ax'+by'$;

2° Il existe au plus n éléments linéairement indépendants, satisfaisant à une même équation d'ordre n:

$$a_n x^{(n)}+\ldots+a_0 x=0 \qquad (a_n\neq 0).$$

3. En posant

$$Lx=a_n x^{(n)}+\ldots+a_0 x,$$

*) Les théorèmes présentés dans cette note concernent un espace linéaire quelconque. Pour un anneau linéaire, des théorèmes à peu près analogues ont été démontrés récemment, moyennant la méthode des déterminants, par M. Kowalski dans un travail qui n'a pas encore été publié.

on a évidemment

$$L(ax+by)=aLx+bLy.$$

Si, pour un élément $x_0 \in \mathcal{E}$, on a $Lx_0=0$, cet élément est indéfiniment dérivable et l'on a $(Lx_0)^{(m)}=L(x_0^{(m)})=0$ pour tout $m=1,2,\ldots$ Si l'une des expressions $L_1x,\ldots,L_2x$ s'annule pour $x=x_0$, il en est de même de leur produit (superposition) $Lx=L_1x\ldots L_nx$.

3. Les deux propositions suivantes ne sont basées que sur l'axiome 1°.

Si les équations $L_1x=0$ et $L_2x=0$ n'ont pas de solution commune et si $L_1x_i=0$ pour $i=1,\ldots,m$, $L_2y_i=0$ pour $i=1,\ldots,n$, chacun des systèmes $x_1,\ldots,x_m$ et $y_1,\ldots,y_n$ étant linéairement indépendant, alors le système de $m+n$ éléments $x_1,\ldots,x_m$, $y_1,\ldots,y_n$ est linéairement indépendant.

Si les équations $L_1x=0$ et $L_2x=0$ ont exactement m et n solutions linéairement indépendantes, l'équation $L_1L_2x=0$ a au plus $m+n$ solutions linéairement indépendantes.

La dernière proposition ne peut pas être renforcée, en affirmant que l'équation $L_1L_2x=0$ a exactement $m+n$ solutions linéairement indépendantes, même lorsqu'on admet les deux axiomes 1° et 2°.

4. L'équation $Lx=0$ est irréductible, lorsque son polynôme caractéristique $a_n\xi^n+\ldots+a_0$ est irréductible. En admettant les deux axiomes 1° et 2°, on a la proposition suivante:

L'équation irréductible $Lx=0$ d'ordre n a exactement n solutions linéairement indépendantes, ou bien n'en a aucune (c'est-à-dire sa seule solution est nulle).

5. Introduisons maintenant dans l'espace $\mathcal{E}$ une nouvelle opération faisant correspondre un élément $tx \in \mathcal{E}$ à tout élément $x \in \mathcal{E}$ de manière que

3° $t(ax+by)=a\cdot tx+b\cdot ty$;

4° Si x' existe, on a $(tx)'=tx'+x$.

Les axiomes 1°, 3° et 4° étant remplis, on a la proposition suivante:

Si Lx s'annule pour $x=x_0$, alors L^ix s'annule pour $x=t^ix_0$, où $i=0,\ldots,n-1$.

6. Voici le résultat principal, basé sur les quatre axiomes 1°—4°.

Si $x_1,\ldots,x_n$ sont des solutions linéairement indépendantes de l'équation $Lx=0$, irréductible et d'ordre n, l'équation $L^mx=0$ (d'ordre mn) a exactement mn solutions linéairement indépendantes: t^ix_j ($i=0,\ldots,m-1$; $j=1\ldots,n$).

7. Les théorèmes précédents permettent de réduire la résolution de toute équation $Lx=0$ à des équations irréductibles. En effet, il suffit de décomposer Lx en facteurs irréductibles

$$Lx=L_1^{k_1}\ldots L_m^{k_m}x;$$

on peut arranger que les premières p équations irréductibles,

$$L_1 x = 0, \ldots, L_p x = 0 \tag{1}$$

aient des solutions non nulles et que les équations restantes,

$$L_{p+1} x = 0, \ldots, L_m x = 0,$$

en soient dépourvues. L'ordre l_i de L_i est, pour les équations (1), le nombre des solutions linéairement indépendantes. Les solutions $L_i^{k_i} x = 0$ peuvent être trouvées, pour $i = 1, \ldots, p$, moyennant le théorème du paragraphe **6**. Pour $i = p+1, \ldots, m$, ces équations n'ont pas d'autre solution que 0. L'ensemble de toutes les solutions des équations $L_i^{k_i} x = 0$ $(i = 1, \ldots, p)$ constitue $k_1 l_1 + \ldots + k_p l_p$ solutions linéairement indépendantes de $Lx = 0$. Ce sont toutes les solutions linéairement indépendantes.

8. Cette méthode peut être appliquée, en particulier, pour résoudre les équations différentielles opérationnelles,

$$a_n x^{(n)}(\lambda) + a_{n-1} x^{(n-1)}(\lambda) + \ldots + a_0 x(\lambda) = 0,$$

où $a_n, \ldots, a_0$ sont des opérateurs [2]. On obtient ainsi un théorème beaucoup plus fort que celui d'un de mes travaux antérieurs [2]; de plus, on évite ainsi de pénibles calculs avec des déterminants (voir [2], p. 229-232, et [5]).

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OUVRAGES CITÉS

[1] S. Drobot et J. G. Mikusiński, *Sur l'unicité des solutions de quelques équations différentielles dans les espaces abstraits*, Studia Math. **11** (1950), 38.

[2] J. G. Mikusiński, *Sur les équations différentielles du calcul opératoire et leurs applications aux équations aux dérivées partielles*. Studia Math. **12** (1951), 227.

[3] — *Sur les fondements du calcul opératoire*, Studia Math. **12** (1950), 41.

[4] — *Sur l'unicité des solutions de quelques équations différentielles dans les espaces abstraits*, Ann. Soc. Polon. Math. **22** (1949), 157.

[5] — *Sur un déterminant*, ibidem **25** (1952), 27.

[6] — *Sur la dérivée algébraique*, Fund. Math. **40** (1953), 99.

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THEORETICAL PHYSICS

Polarization of Nucleons from Photonuclear Reactions

by

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Presented by L. INFELD on December 19, 1955

In recent years, many authors have investigated the polarization of nucleons scattered elastically from nuclei or produced in certain nuclear reactions of the stripping type. The aim of the present paper is to give a method of calculating the degree of polarization of nucleons produced in photonuclear reactions, together with certain applications.

On applying the time-dependent theory of perturbations to the calculation of the cross-section for photodisintegration [1], [2], the wave function of the system nucleon-nucleus $\psi(\vec{r},t)$ is represented by a superposition of the stationary energy eigenfunctions. The photonucleon is allowed to interact with the final nucleus through a potential containing in general a spin-orbit term. Then J^2, L^2, M_J, σ^2 are constants of the motion. The energy eigenfunctions $\psi_E(L, J, M_J)$ are then simultaneous eigenfunctions of these operators. Thus we may write:

$$\psi(\vec{r},t)=\Big(\sum_{E'}+\int dE'\Big)\Big\{\sum_{L,J,M_J} a_{E'}(L,J,M_J;t)\psi_{E'}(L,J,M_J)\Big\}, \tag{1}$$

where $\psi_{E'}(L,J,M_J)=R_{LJ}(E')Y(L,J,M_J)\,e^{\frac{-iE't}{\hbar}}$, $R_{LJ}(E')$ depends on r and $Y(L,J,M_J)$ is the eigenfunction of the set of the commuting operators J^2, L^2, M_J, σ^2. In view of the asymptotical form of $R_{LJ}(E')\sim N(E')r^{-1}\sin(k'r-\frac{1}{2}L\pi+\eta_{ls})$ *), we obtain in analogy to [2]:

$$\psi(\vec{r},t)\sim r^{-1}\,e^{ikr-i\frac{Et}{\hbar}}\,N(E)\sum_{L,J,M_J}{}' e^{i\left(\eta_{LJ}-\frac{1}{2}L\pi\right)}Y(L,J,M_J)\,H'^{L,J,M_J}_{l,j,m_j}, \tag{2}$$

where $N(E)$ is a factor dependent only on energy and

$$H'^{L,J,M_J}_{l,j,m_j}=\int R^*_{LJ}Y^*(L,J,M_J)H'R_{lj}Y(l,j,m_j)\,d\tau,$$

*) Valid for neutrons; the generalization for protons is straightforward.

H' being the interaction of the nucleon with the electromagnetic field, $R_{lj}Y(l,j,m_j)$ and $R_{LJ}Y(L,J,M_J)$ are single-particle model wave functions of the initial and final states.

The polarization vector is

$$\vec{P}=\frac{\{<f\chi\,\vec{\sigma}_n|f\chi>\}_{av}}{\{<f\chi|f\chi>\}_{av}},$$

where $f\chi$ is the scattering amplitude obtained from (2) and equal to $f_{+1/2}\chi_{+1/2}(\sigma_n)+f_{-1/2}\chi_{-1/2}(\sigma_n)$; $\chi_{\mu_n}(\sigma_n)$ is the neutron spin function and

$$f_{\mu_n}=\sum_{L,J,M_J} i^{-L}e^{i\eta_{LJ}}C_{L^1/_2}(J,M_J;M_J-\mu_n,\mu_n)Y_{L,M_J-\mu_n}(\bar{\vartheta},\bar{\varphi})H'^{L,J,M_J}_{e,j,m_j}, \tag{3}$$

where $C_{L^1/_2}(J,M_J;M_J-\mu_n\cdot\mu_n)$ is the vector addition coefficient and $\bar{\vartheta}$, $\bar{\varphi}$ are the polar angles of the vector $\vec{k}$ of the outgoing neutron; the symbol $\{\}_{av}$ denotes averaging over direction of γ quantum polarization and magnetic quantum numbers of the initial nucleus.

We confine ourselves to the dipole electric approximation. Then H' is proportional to $\vec{e}\,\vec{r}=r\cos\alpha$, where $\vec{e}$ is the polarization vector of the incident photon. The system of reference is so chosen that the xy-plane is identical with the reaction plane, and the z-axis with the direction of the vector $\vec{k}\times\vec{k}_0$, where $\vec{k}_0$ is the wave vector of the photon. Then

$$\vec{e}\,\vec{r}=r\cos\alpha=r(\cos\vartheta\cos\Phi+\sin\vartheta\sin\Phi\cos\varphi)=z\cos\Phi+x\sin\Phi. \tag{4}$$

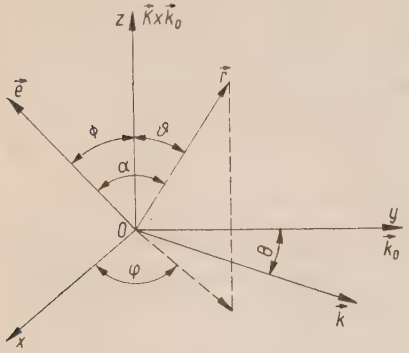

Fig. 1. Geometrical situation of the problem

The significance of the various angles involved is indicated in Fig. 1. With the quantization axis chosen ($\bar{\vartheta}=\pi/2$, $\bar{\varphi}=\pi/2-\theta$, where θ is the scattering angle, see Fig. 1), it is easy to ascertain that P_x and P_y vanish. The degree of polarization is

$$P=P_z=\frac{\{|f_{+1/2}|^2-|f_{-1/2}|^2\}_{av}}{\{|f_{+1/2}|^2+f_{-1/2}|^2\}_{av}}. \tag{5}$$

In the present case we may write:

$$f_{\mu_n}=f^z_{\mu_n}\cos\Phi+f^x_{\mu_n}\sin\Phi. \tag{6}$$

Averaging over the directions of the photon polarization gives

$$\{|f_{\mu_n}|^2\}_{av}=\frac{1}{2(2j+1)}\sum_{mj=-j}^{j}(|f^z_{\mu_n}|^2+|f^x_{\mu_n}|^2). \tag{7}$$

Since $|f^z_{\mu_n}|^2$ give a zero contribution to the numerator of the expression for P, the polarization becomes

$$P=\frac{\sum\limits_{mj}(|f^x_{+1/2}|^2-|f^x_{-1/2}|^2)}{\sum\limits_{\mu_n}\sum\limits_{mj}(|f^z_{\mu_n}|^2+|f^x_{\mu_n}|^2)}. \tag{8}$$

At first, this method may be applied to the calculation of the polarization of low and medium energy *) photoneutrons from the $^9\mathrm{Be}(\gamma,n)^8\mathrm{Be}$ reaction. The single-particle model of Guth and Mullin [3] may be assumed for the ^{9}Be nucleus, its ground state being $P_{3/2}$. The assumption that the final neutron interacts with the residual core through the potential containing a spin-orbit coupling, after performing the angular integrations and calculating the coefficients, gives

$$P=\frac{i[AB^*-A^*B+\frac{1}{3}(C^*D-CD^*)]\sin 2\theta}{I+(AA^*+BB^*)+\frac{1}{3}(CC^*+DD^*)+[AB^*+A^*B+\frac{1}{3}(CD^*+C^*D)]\times\cos 2\theta}, \tag{9}$$

where

$$I=\tfrac{4}{3}\,|R_{1/2}+\tfrac{1}{10}R_{3/2}+\tfrac{9}{10}R_{5/2}|^2+\tfrac{9}{25}\,|R_{3/2}-R_{5/2}|^2,$$

$$A=R_{1/2}-\tfrac{1}{5}R_{3/2}-\tfrac{3}{10}R_{5/2}, \qquad B=\tfrac{3}{2}R_{5/2},$$

$$C=R_{1/2}+\tfrac{2}{5}R_{3/2}-\tfrac{9}{10}R_{5/2}, \qquad D=\tfrac{3}{5}R_{3/2}+\tfrac{9}{10}R_{5/2};$$

$$R_J=e^{i\eta_{LJ}}\int_0^\infty R^*_{LJ}R_{1^3/_2}r^3\,dv.$$

If the spin-orbit coupling is equal zero then $R_{3/2}=R_{5/2}$ and $A=C$, $B=D$. This applies to the case of the interactions of Guth and Mullin (square well-depths are $V_{1^3/_2}=12.16$ MeV, $V_{0^1/_2}=V_{2^3/_2}=V_{2^5/_2}=3$ MeV). In this case

$$P=\frac{2R_{PS}R_{PD}\sin(\eta_2-\eta_0)\sin 2\theta}{A+B\cos 2\theta},$$

$$A=\tfrac{4}{3}R^2_{PS}+\tfrac{7}{3}R^2_{PD}+\tfrac{2}{3}R_{PS}P_{PD}\cos(\eta_2-\eta_0), \tag{10}$$

$$B=-R^2_{PD}+2R_{PS}R_{PD}\cos(\eta_2-\eta_0)),$$

where η_0,η_2 are the S and D state phase shifts defined in [3] as δ_0 and δ_2,

$$R_{PS}=\int_0^\infty R^*_{0^1/_2}R_{1^3/_2}r^3\,dr, \qquad R_{PD}=\int_0^\infty R^*_{2^3/_2}R_{1^3/_2}r^3\,dr=\int_0^\infty R^*_{2^5/_2}R_{1^3/_2}r^3dr.$$

P was calculated for certain energies of the incident γ quantum $\hbar\omega$ close above the threshold of the reaction = 1.63 MeV **). For these energies

*) The γ quantum energy in the range of validity of the model used.

**) According to [3]. However, the last experimental value is 1.66 MeV. This difference has little influence on the numerical results.

the percentage dependence of P on θ is shown graphically in Fig. 2. The maximum P is considerable only in the energy range for which both R_{PS} and R_{PD} are simultaneously considerable, that is, in a width of about 1 MeV near the threshold of the reaction. For larger energies of the γ quantum — P is negligible. From Fig. 2 it is seen that $P_{\max}(\approx -45\%$ for $\hbar\omega = 2$ MeV) comes in near $\theta \approx 40^0$.

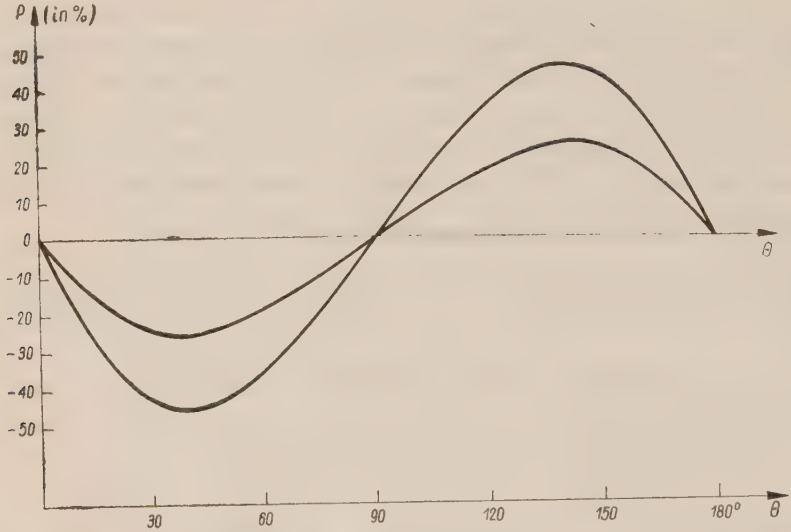

Fig. 2. Polarization of neutrons from the $^9\mathrm{Be}(\gamma, n)\,^8\mathrm{Be}$ reaction (in %) as function of the scattering angle θ for $\hbar\omega = 2$ and 3 MeV. The curve for $\hbar\omega = 2.62$ MeV as found almost coincident with that for $\hbar\omega = 2$ MeV. The curve for which the absolute value of polarization is greater is the $\hbar\omega = 2$ MeV curve; that for which polarization is smaller applies to $\hbar\omega = 3$ MeV

It is interesting to note here that $P \neq 0$ comes in for the central interaction V_{LJ}. But if the initial state of a particular nucleus is $S_{1/2}$ there is obtained for an interaction with the spin-orbit coupling the polarization

$$P = \frac{i(FG^* - F^*G)\sin 2\theta}{Z + (FF^* + GG^*) + (F^*G + FG^*)\cos 2\theta}, \tag{11}$$

where

$$F = -\bar{R}_{3/2} - 2\bar{R}_{1/2}, \qquad G = 3\bar{R}_{3/2}, \qquad Z = 4\,|\bar{R}_{3/2} - \bar{R}_{1/2}|,$$

$$\bar{R}_J = e^{i\eta_{LJ}} \int_0^\infty R^*_{LJ} R_{0^1/_2} r^3\, dr.$$

This result may, for instance, be applied to the calculation of polarization of photoneutrons from the ^{29}Si nucleus close to the threshold of the $^{29}\mathrm{Si}(\gamma, n)$ reaction. From Eq. (12) it follows that if the spin-orbit coupling vanishes ($\bar{R}_{3/2} = \bar{R}_{1/2}$), then $P \equiv 0$.

The experimental investigation of P seems to be important since it might make available valuable information concerning the types of interactions for certain simple nuclei such as ^{2}H, ^{6}Li, ^{7}Li, ^{9}Be, for which two body models may be applied. The calculations for the $^2\mathrm{H}(\gamma, n)$ reaction for the examination of the $\vec{L}\cdot\vec{\sigma}$ coupling, e. g. as proposed by Case and Pais [4] and used by Austern, are in preparation.

It is hoped to develop the method to cover the assumption that the outgoing nucleon interacts with the residual core through the "clouded crystal ball" potential of Feshbach et al. [6] modified by Adair et al. [7] by an additional spin-orbit term.

Our thanks are extended to Professors L. Infeld and H. Niewodniczański for their kind interest.

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REFERENCES

[1] H. A. Bethe, Handbuch der Physik, **24** (1931), 482.
[2] — Annalen der Physik, **4** (1930), 443.
[3] E. Guth, C. Mullin, Phys. Rev. **67** (1949), 234.
[4] K. M. Case, A. Pais, Phys. Rev. **80** (1950), 203.
[5] N. Austern, Phys. Rev. **85** (1952), 283.
[6] H. Feshbach, C. E. Porter, V. F. Weisskopf, Phys. Rev. **96** (1954), 448.
[7] P. Adair, P. Fields, C. Darden, Phys. Rev. **96** (1954), 503.

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On the Weizsäcker Correction in the Statistical Theory of Molecular Structure

by

W. KOŁOS

Presented by L. INFELD on January 11, 1956

A molecular orbital or valence bond calculation of the bond energy of a molecule or a bond requires that the hybridization of the atomic orbitals be known. However, the determination of the hybridization parameters is, even for molecules built up from light atoms, very difficult, and for heavy atoms, where the d, f orbitals are to be taken into account, almost impossible. Therefore, it seems that the statistical method might well be suited for this purpose since, when using this method, we do not need to know anything about the hybridization of the orbitals.

However, Sheldon [5] has recently used the statistical method to calculate the bond energy of the N_2 molecule, and has shown that this method does not give a stable molecule. From the results of his calculations, he was able to conclude quite generally that the statistical method cannot account for molecular binding. It must be added, however, that this is true only in the approximation used by Sheldon.

Sheldon has solved the Thomas-Fermi equation, exchange including, for the nitrogen molecule, but it is well known that in atomic problems the statistical method is fully successful only when the Weizsäcker [6] and Gombás [1] corrections are taken into account. It has not been possible up to now to use the Gombás correction in problems of no spherical symmetry, but we can estimate the influence of the Weizsäcker correction on the bond energy of the molecule.

The energy of a statistical atom model disregarding the Gombás correction, is given by the well known expression *)

$$E = 2\varkappa_0 \int \varrho^{5/3} dv - \int \frac{ze^2}{r} \varrho \, dv + \frac{1}{2} e^2 \int \frac{\varrho(\vec{r}) \, \varrho(\vec{r}')}{|\vec{r} - \vec{r}'|} dv \, dv' - \tag{1}$$

$$- \varkappa_a \int \varrho^{4/3} dv + \varkappa_\omega \int \frac{(\operatorname{grad} \varrho)^2}{\varrho} dv .$$

*) For the notation and more details concerning this method see [1] and [2].

If we want to use this theory to calculate the energy of a statistical molecule model we have to include in the total energy (1), the interaction energy of the nuclei, and to take into account the interaction energy of the electrons with all nuclei.

We have calculated the energy of the N_2 molecule (for the nuclei separation $R = 1.096$ Å) assuming for the density distribution of the electrons a function of the form

$$\varrho = \frac{N_M}{A} e^{-\lambda(r_a + r_b)}[1 + c(r_a - r_b)^2]^3, \tag{2}$$

where λ and c are variational parameters, N_M is the number of electrons in the molecule, and A can be determined from the condition

$$\int \varrho \, dv = N_M e. \tag{3}$$

We repeated these calculations neglecting the Weizsäcker correction, i. e. the last term in (1).

In the same approximation, we have also calculated the energy of an N atom, assuming for the electron density distribution a function of the form

$$\varrho = \frac{N_A}{A} e^{-\lambda r}(1 + cr^2)^3, \tag{4}$$

where N_A is the number of electrons in the atom.

The energy terms in (1) for the molecule, with ϱ given by (2), can be easily reduced to the well known integrals $A_n(a)$ and $W_\tau(m, n; a)$. The method of calculation of these integrals has been given by Kotani, Amemiya and Simose [1] who have also tabulated these integrals for various values of the parameter a.

The results of our calculations and the values of the variational parameters corresponding to the energy minima are given in Table I.

TABLE I

	N_2 molecule			N atom		
	λR	cR^2	$-E$	λ	c	$-2E$
With the Weizsäcker correction	3.35	1.5	60.6	5.5	0.66	65.0
Without the Weizsäcker correction	4.70	5.1	86.4	8.5	1.80	98.4
Sheldon's results			154.80			155.89
Experimental [3]						109.16

In neither case did we get any binding. This is not surprising, because from the results for the N atom we see that our approximation

is a rough one, and to get better results we should have to introduce more variational parameters in (2) and (4). However, we see that including the Weizsäcker correction we got a much smaller repulsion energy than without this correction. So we may expect that if the numerical calculations were performed as exactly as Sheldon's, it would be possible to get a binding.

I would like to express my sincere thanks to Professor P. Gombás for valuable suggestions and discussions of the problem, and to the Hungarian Academy of Sciences for enabling me to do this work partly in the Institute of Physics at the University of Technical Sciences in Budapest.

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REFERENCES

[1] P. Gombás, Acta Phys. Hung. **3** (1953), 105, 127.

[2] — *Die statistische Theorie des Atoms und ihre Anwendungen*, Wien, Springer Verlag, 1949.

[3] H. Hellman, *Einführung in die Quantenchemie*, Franz Deuticke, Leipzig und Wien, 1937, p. 16.

[4] M. Kotani, A. Amemiya and T. Simose, Proc. Phys. Math. Soc. Japan **20** (1953), Extra No. 1.

[5] J. W. Sheldon, Phys. Rev. **99** (1955), 1291.

[6] C. F. v. Weizsäcker, Z. Phys. **96** (1935), 431.

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CHEMISTRY

On Nitration of Salicylhydroxamic Acid

by

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Communicated by T. URBAŃSKI at the meeting of November 28, 1955

Preliminary experiments on nitration of salicylhydroxamic acid carried out by J. Nowak and one of us (T. U.) [1] led to the conclusion that, depending on the conditions of nitration, a high yield of picric acid can be obtained.

More detailed experiments were carried out by the authors of the present paper and produced the following conclusion:

When salicylhydroxamic acid was nitrated with nitric acid ($d=1.50$) alone at a temperature below 60°, 5-nitrosalicylic acid was formed with a total yield of 71% of theoretical. When the nitration was carried out with nitric acid ($d=1.50$) at a temperature above 65°, picric acid with a yield of c. 60% was obtained.

It is well known that picric acid (and 5-nitrosalicylic acid) can be formed by nitration of salicylic acid [2], [3]. The former is most probably produced by decarboxylation, which occurs simultaneously with the nitration. We repeated the experiments of Marchand and Hübner. We found that in the conditions of experiments of nitration of salicylhydroxamic acid with nitric acid leading to the formation of picric acid (i. e. above 65°), the yield of picric acid and of 5-nitrosalicylic acid from salicylic acid was c. 11% and 7% respectively.

This led us to a conclusion that the formation of picric acid from salicylhydroxamic acid cannot be explained solely as the result of an intermediate formation of salicylic acid and nitration, and subsequent decarboxylation.

However, we found that under action of nitric acid diluted with ether to a concentration of c. 40 per cent HNO_3 at c. −15°, a hydrolysis of salicylhydroxamic acid to salicylic acid occurred with a yield of 83%.

This seems to demonstrate that the first stage of nitration of salicylhydroxamic acid may partly consist in converting it into salicylic acid. The possibility of a reaction similar to the Lossen rearrangement was

considered. No experimental support for such a possibility has, however, been found so far.

When salicylhydroxamic acid was nitrated with a mixture of nitric acid (d=1.40) and sulphuric acid in the proportion 35 : 50 at a temperature of c. 95°, the main product of the reaction was 3.5-dinitrosalicylic acid with a yield of c. 20%.

When the reaction was carried out with a mixture of nitric acid (d=1.50) and acetic anhydride in the proportion 1 : 1, at temperatures of 20°-60°, two products — 5-nitrosalicylic and 3.5-dinitrosalicylic acids — were formed.

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REFERENCES

[1] T. Urbański and J. Nowak, unpublished paper (1952).
[2] R. F. Marchand, J. pract. Chem. (1), **26** (1843), 397.
[3] H. Hübner, Ann. **195** (1879), 45.

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CHEMISTRY

Hetero-Polyazeotropic Systems I.
System Methanol — *n*-Paraffinic Hydrocarbons

by

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Presented by W. ŚWIĘTOSŁAWSKI on December 3, 1955

Introductory remarks

The possibility of the continuous transition with the change of pressure from hetero- to homoazeotropes in two-component systems, was considered some years ago by Świętosławski [1]. He indicated the close association of these two kinds of systems.

It seemed to be interesting to prove also the transition from heteroazeotropy into homoazeotropy at constant pressure. In this case, it is necessary to examine systems (A, H_i) which are formed by an azeotropic agent A with the representative of a homologous series $H_1, H_2, \ldots, H_i$, replacing one another in the binary system. A series of boiling temperature isobars of such a series of two-component systems (A, H_i) were called by Świętosławski [2] a polyazeotropic system. Considering polyazeotropic systems of the type (A, H_i), characterized by limited mutual miscibility of A and H_i, we can find three typical cases as discussed below.

Classification of hetero-polyazeotropic systems

1. Let A be a substance composed of polar molecules of a small number of carbon atoms, and series H_i representing the homologues of paraffins. In such a case, the greater the number of carbon atoms in the molecule, and the greater the increase in the boiling temperature of the homologue, the greater are the differences in the structure and the size of the molecules under consideration. Consequently, an increase should be expected in the upper critical solution temperature T_c in the series of systems formed. In a particular case, the region of limited mutual miscibility may lie within the limits of the azeotropic range $Z_A(H_i)$ of agent A. Depending on whether the homologue H_i will show limited miscibility with A, we shall observe either all possible types of phase

equilibrium, or certain types only. The analysis of all the cases will not be considered here. Up to date, the miscibility of a few two-component series has been investigated [4], [5]. An increase of the critical solution temperature was always observed in the systems: methanol — n-paraffins, acetonitryle — n-paraffins, ethylenediamine — n-paraffins, nitromethane — n-paraffins, acetic acid — n-paraffins, and formic acid — n-paraffins.

2. There exist polyazeotropic systems showing phenomena opposite to those mentioned above. The difference in the structures and in the size of molecules diminishes with the increase in the boiling temperature of representative H_i. Consequently, the critical solution temperature decreases and a gradual transition from hetero- to homoazeotropy is observed, for instance in the case of the systems formed by n-undecane with alcohols, or n-undecane with aliphatic fatty acids.

3. A case may also be found in which agent A does not differ very much in structure and molecular size as compared with the homologues $H_1, H_2, \ldots, H_i$. If so, first a decrease in the differences between A and H_i may be expected in the polyazeotropic system (A, H_i), then further an increase in those differences may be observed; consequently the critical solution temperature may decrease in some cases, and then increase in others. When dealing with such systems, we may expect that the critical solution temperatures should lie below the azeotropic range $Z_A(H_i)$, at least when the system is kept under normal pressure.

It has to be emphasized that these phenomena may easily be found in ternary hetero- and homoazeotropic systems. This is due to the third component, which exerts its influence on the miscibility of the remaining two components. In such cases, changes occur not only in the upper, but frequently in the lower critical solution temperature.

The system composed of methanol and the respectives of n-paraffins, comes within the first group of the classification mentioned above.

Experimental work. Preparation of substances

Methanol was dehydrated and purified by rectification in the presence of metallic sodium. The paraffinic hydrocarbons were isolated from the mixture obtained by Fischer-Tropsch synthesis. The unsaturated hydrocarbons present in that mixture were removed by repeated shaking with sulphuric acid.

Investigation method

The boiling temperature isobars were determined by using a single stage Świętosławski ebulliometer. A second ebulliometer, used for pressure control, was filled with methanol to obtain strictly comparative conditions for reducing the boiling temperature to a constant pressure.

Special care was taken to prevent contamination with moisture. The critical solution temperatures were determined by the conventional method.

Experimental results

In Fig. 1 are shown the boiling temperature isobars at 760 mm. Hg of two-component mixtures of methanol with *n*-paraffinic hydrocarbons. Points *Az* represent the composition (in mole percentages) of the respective azeotropic mixture. The horizontal sections of the isobars correspond to equilibria established between the two liquid phases and the vapour phase.

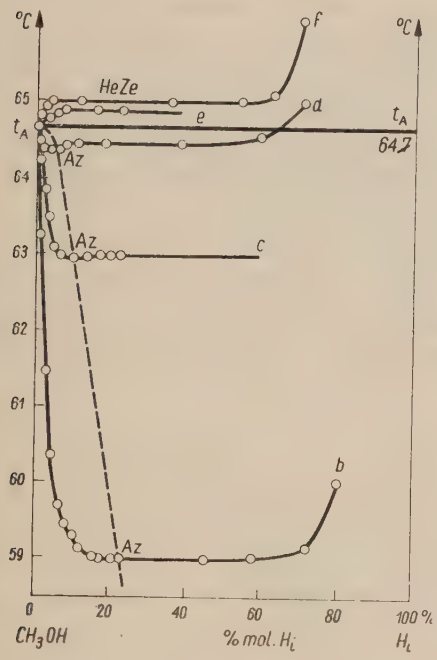

Fig. 1

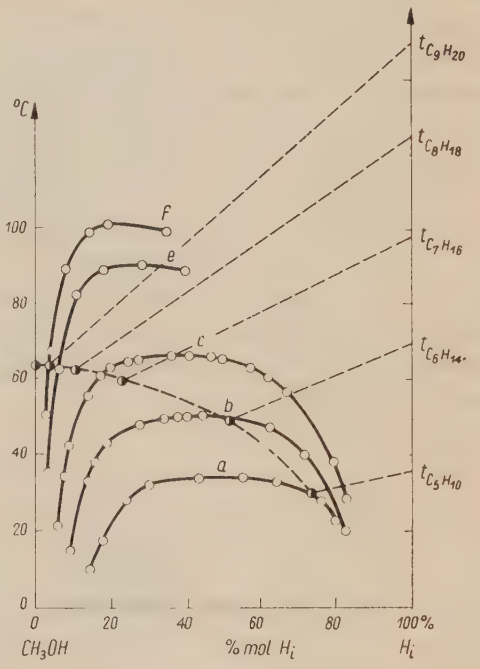

Fig. 2

Fig. 1. Series of boiling temperature isobars of methanol (*A*) with the hydrocarbons (H_i): (*b*) *n*-heptane, (*c*) *n*-octane, (*d*) *n*-nonane, (*e*) *n*-decane, (*f*) *n*-undecane

Fig. 2 Series of mutual miscibility curves of two-component systems of methanol with hydrocarbons: (*a*) *n*-hexane, (*b*) *n*-heptane, (*c*) *n*-octane, (*e*) *n*-decane, (*f*) *n*-undecane. The compositions are expressed in mol per cent of H_i

The miscibility curves of methanol with *n*-hexane, *n*-heptane and *n*-octane (Fig. 2), were determined by Sieg [6], those with *n*-decane and *n*-undecane — by ourselves. The temperature scale on Fig. 2 is considerably reduced in comparison with Fig. 1. The dotted line on Fig. 2 represents the change of the azeotropic composition of the system under consideration. Important data are given in Table I; a full collection of data and more detailed description of the system examined will appear in Roczniki Chemii.

Discussion

Although the isobars of systems formed by *n*-butane, *n*-pentane and *n*-hexane with methanol were not investigated, it is quite certain that these systems are homoazeotropic, because the critical solution temperatures lie considerably below the boiling temperatures of the corresponding azeotropes. The system formed by *n*-heptane and methanol is homoazeotropic Because of the vicinity of the critical solution temperature, the corresponding isobar is very flat. The systems: *n*-octane — methanol and *n*-nonane — methanol are both homoazeotropic and heterozeotropic. The azeotropic points in these systems lie outside the mutual miscibility curve and there exist flat sections on the isobars in the three-phase region. Starting with *n*-decane, the systems are typically heterozeotropic. The zeotropic temperature increase in relation to the boiling temperature of methanol 64.7°C is small, not exceeding several decimal fractions of a degree. In the propane — methanol system, the heterozeotropy may be assumed, notwithstanding that the given value of the critical solution temperature is not quite certain [5].

In the series investigated, an increase is observed in the critical solution temperature, together, with the characteristic assymetry in the location of the critical point. The amount of methanol at the critical composition increases with the increase of the boiling temperature of the hydrocarbon.

TABLE I

Characteristics of the systems methanol—*n*-paraffins

System	Boiling temperature of the hydrocarbon °C.	Critical solution temperature °C.	Boiling temperature of the azeotrope °C.	Type
methanol — *n*-propane	−42	−32 [5]	?	?
methanol — *n*-butane	0—	−17 [5]	?	homoazeotrope
methanol — *n*-pentane	36.07	14	30.4 [7]	„ „
methanol — *n*-hexane	68.74	34.8	49.5 [7]	„ „
methanol — *n*-heptane	98.4	51.0	59.1	„ „
methanol — *n*-octane	125.6	66.7	63.0	homoazeotrope-heterozeotrope
methanol — *n*-nonane	150.7	?	64.4	„ „
methanol — *n*-decane	174.2	91	64.93	heterozeotrope
methanol — *n*-undecane	195.8	103	65.05	„ „

The authors express their thanks to Professor W. Świętosławski for numerous suggestions.

Summary

1. A series of boiling temperature isobars of two-component mixtures of methanol with *n*-heptane, *n*-octane, *n*-nonane, *n*-decane and *n*-undecane was examined. Some regularities were found in the gradual transition from homoazeotropy to heterozeotropy.

2. Critical solution temperatures were determined for mixtures of *n*-decane and *n*-undecane with methanol.

3. A classification was given for heterogeneous systems of the type (A, H_i), where H_i is one of the representatives of the homologous series of paraffins.

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REFERENCES

[1] W. Świętosławski, Roczniki Chem. **13** (1933), 127.
[2] — Bull. Soc. Chim. Belges, 62 (1953), 10; *Azeotropia i poliazeotropia*, (in press).
[3] — *Ebulliometric Measurements*, New York 1945.
[4] A. Seidel, *Solubility of organic compounds*, Vol. II, New York 1941.
[5] A. Farkas, *Physical chemistry of hydrocarbons*, Vol. I, New York 1950, p. 287.
[6] L. Sieg, Chem. Ing. Technik, **5** (1951), 112.
[7] L. Horsley, *Tables of azeotropes and nonazeotropes*, 1952.

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CHEMISTRY

Reduced Equation Characterizing Certain Families of Binary Azeotropic Systems

by

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Introductory remarks

The terms used in this paper — azeotropic range $Z_A(H)=T_{H_k}-T_{H_e}$ and the lower and upper portion $Z_d=T_A-T_{H_e}$ and $Z_g=T_{H_k}-T_{H_e}$, and a series of positive binary azeotropes: $(AH_e)...(AH_i)...(AH_k)$ — were formulated in previous papers [1], [2]. In addition, the following recently developed relations [3] are used as the basis for transforming them into the reduced equations.

(1a) $$\delta_A=Z_d x_{H_i}^2$$

(1b) $$\delta_{H_i}=Z_g x_A^2$$

(2a) $$x_{H_i}=\frac{1}{2}+\frac{\Delta_A}{Z}$$

(2b) $$x_A=\frac{1}{2}+\frac{\Delta_{H_i}}{Z}$$

(3a) $$\Delta_A=-\tfrac{1}{2}Z+\sqrt{2Z}\,\sqrt{\delta_A}$$

(3b) $$\Delta_{H_i}=-\tfrac{1}{2}Z+\sqrt{2Z}\,\sqrt{\delta_{H_i}}$$

where:

(4) $$Z_d=\frac{A_{AH}}{\Delta S_A};\qquad Z_g=\frac{A_{AH}}{\Delta S_H};$$

(5) $$\delta_A=T_A-T_{Az};\qquad \delta_{H_i}=T_{H_i}-T_{Az};$$

(6) $$\Delta_A=T_A-T_{H_i};\qquad \Delta_{H_i}=T_{H_i}-T_A;$$

The symbols found in the equations (1), (2), (3) represent: x_A and x_{H_i} the mole fractions of A and H_i in the binary azeotrope (AH_i); δ_A, δ_{H_i} the azeotropic depressions in relation to agent A and homologue H_i, respectively; Δ_A, Δ_{H_i} are equal to the boiling temperature differences of the components. Values δ_A, δ_{H_i} and Δ_A, Δ_{H_i} are directly defined by (5), (6). Value A_{AH} is the empirical constant of regular solutions for the entire series of azeotropes formed by A with the representatives of series (H). This constant does not depend either on temperature or on the concentration of all of the azeotropes belonging to the series (AH_i). ΔS_A and ΔS_H are the evaporation entropies. The equations (2), (3) have been obtained [3] by assuming that $\Delta S_A = \Delta S_H$ and consequently $Z_d = Z_g = \frac{1}{2}Z$ (symmetrical azeotropic range). For the sake of simplicity, the symbol $Z_A(H)$ is replaced below by Z. It must be remembered that our basic equations (1), (2), (3) are derived under the assumption that all mixtures of the type (AH_i) are regular solutions. If there are substances with a very distinctly pronounced polarity and inclination to form associated complexes and hydrogen bonds, then some larger deviations from the relations expressed by the equations (1), (2), (3) are to be expected. From this point of view, it seems reasonable to accept as standard the homologous series of normal hydrocarbons, so as to be able to compare the relations obtained for other series of homologues with those characterizing the normal paraffins.

It should be emphasized that in many cases the value $Z_A(H)$ for the azeotropic range of A in relation to the series (H) of homologues and their isomers, cannot be obtained by direct experiments in view of the lack of representatives H_k and H_e forming the tangent boiling temperature isobars. Nevertheless, the mathematical relation (4) determining the azeotropic range remains valid.

Taking into consideration the equations (1), (2), (3) as they are written above, the value may be calculated as though the homologues forming tangent isobars with agent A really existed. For calculating value Z we must know at least one azeotrope of the series.

Equations with reduced parameters

Let us introduce the following reduced parameters:

$$(7) \qquad \delta_A^{(r)} = \frac{\delta_A}{Z_d}; \quad \delta_A^{(r)} = \frac{\delta_{H_i}}{Z_g}; \quad \Delta_A^{(r)} = \frac{\Delta_A}{\frac{1}{2}Z}; \quad \Delta_{H_i}^{(r)} = \frac{\Delta_{H_i}}{\frac{1}{2}Z}.$$

It is clear that, by dividing the equation (1a) by Z_d, (1b) by Z_g, (3a) and (3b) by $\frac{1}{2}Z$, and substituting $\Delta_A^{(r)}$ into the equation (2a) and $\Delta_{H_i}^{(r)}$

into the equation (2b), the following six equations are obtained:

(8a) $$\delta_A^{(r)} = x_{H_i}^2,$$

(8b) $$\delta_{H_i}^{(r)} = x_A^2,$$

(9a) $$x_{H_i} = \frac{1+\Delta_A^{(r)}}{2},$$

(9b) $$x_A = \frac{1+\Delta_H^{(r)}}{2},$$

(10a) $$\Delta_A^{(r)} = 2\sqrt{\delta_A^{(r)}} - 1,$$

(10b) $$\Delta_{H_i}^{(r)} = 2\sqrt{\delta_{H_i}^{(r)}} - 1.$$

These do not contain any constant values characterizing any particular series of azeotropes. For this reason they express the relations for all such series as fulfill the assumptions on which the equations (1), (2), (3) are based. In Figures 1, 2 and 3, the relations (8a), (9a) and (10a) are graphically presented.

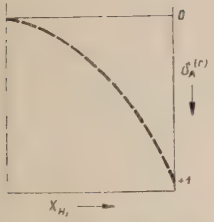

Fig. 1. Reduced azeotropic depression $\delta_A^{(r)}$ plotted against the mole fraction x_{H_i} of the representative H_i according to the equation (8a)

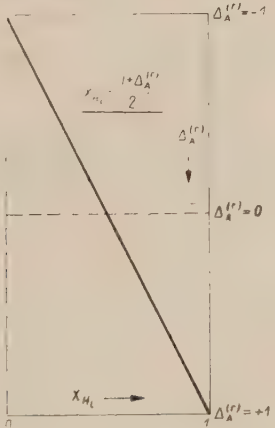

Fig. 2. Reduced difference of the boiling temperatures $\Delta_A^{(r)}$ plotted against the mole fraction x_{H_i} (equation (9a))

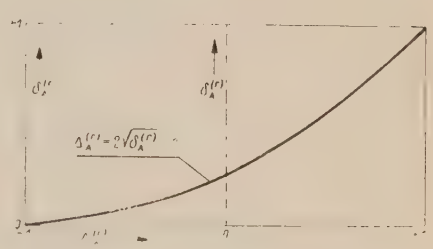

Fig. 3. Reduced depressions $\delta_A^{(r)}$ plotted against the reduced temperature differences $\Delta_A^{(r)}$ (equation (10a))

It is clear that $\delta_A^{(r)}$ undergoes changes from 0 to 1, the limiting values correspond to the tangent azeotropes (AH_k), and $\Delta_A^{(r)}$ changes within the limits from -1 to $+1$ for azeotropes (AH_k) and (AH_e).

In order to prove the applicability of the reduced equation (8a) in Table I, the results of calculation on the basis of experimental data [4] are listed for different binary azeotropes. Eight groups of binary azeo-

tropes (AH_i) have been chosen, formed by eight different azeotropic agents A and by a series of paraffinic, one ring and two ring aromatic hydrocarbons. In Table I, the following numerical values are given: Z_d, Z'_d, δ_A, x_{H_i} and $\delta_A^{(r)}$. All of them except Z'_d are defined above. Symbol Z'_d denotes the value of Z_d, which seems to be nearest to the real one.

In Fig. 4, the location of points $(\delta_A^{(r)}, x_{H_i})$ is shown with the indication of the numbers of the series and of each of the representatives (for instance: III, 4 is the azeotrope formed by isoamyl alcohol and normal octane). Clearly, most of the points lie close to the universal curve $(\delta_A^{(r)} = x_{H_i}^2)$ in spite of a somewhat large variety of physicochemical properties of the azeotropic agent A forming the series of azeotropes.

In order to show typical deviations, azeotropes belonging to series

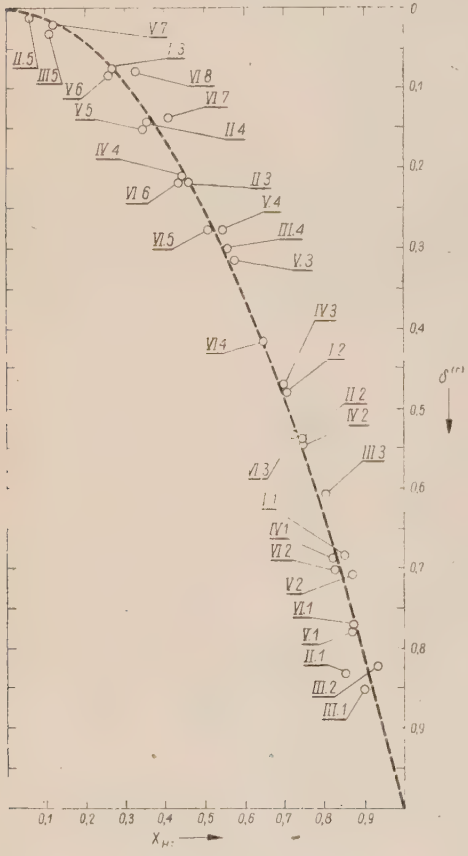

Fig. 4

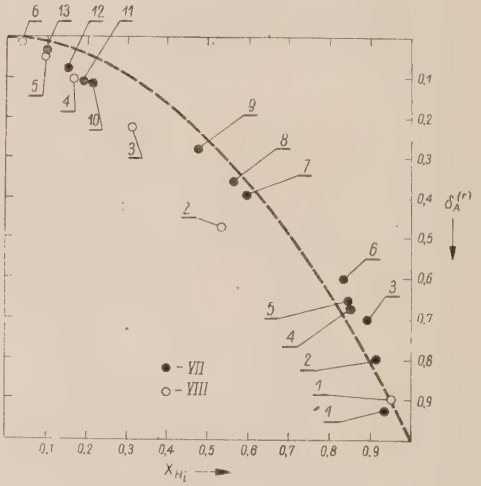

Fig. 5

VII and VIII formed by methanol and acetic acid were chosen. The curve $(\delta_A^{(r)} = x_{H_i}^2)$ and the deviations observed for nineteen azeotropes (VII, 1 – VII, 13 and VIII, 1 – VIII, 6) are shown in Fig. 5. It is obvious that in the examples examined the azeotropic agents are responsible for the deviations observed. Further varieties of positive azeotropes must be examined in order to find a detailed and quantitative explanation of deviations of properties of azeotropes from those of azeotropic mixtures forming regular solutions.

TABLE I

No.	A	No.	Homologue H	Z_d	Z_d'	δ_A	x_H	$\delta_{A(r)}$
I	*n*-propyl formate	1	2, 3-dimethylbutane	34	36.0	24.65	85	0.685
		2	*n*-hexane	35		17.25	70.5	0.479
		3	*n*-heptane	38		2.65	26.5	0.074
II	acetone	1	2-methylbutane	42	45.0	37.45	85	0.832
		2	*n*-pentane	44		24.25	74.5	0.539
		3	2, 3-dimethylbutane	45		9.85	47	0.219
		4	*n*-hexane	50		6.45	36	0.143
		5	*n*-heptane	61		0.3	7	0.067
III	isoamyl alcohol	1	*n*-heptane	41	40.0	34.0	91	0.850
		2	isooctane	38		32.9	93	0.823
		3	2, 4-dimethylhexane	38		24.3	80	0.608
		4	*n*-octane	37		12.1	57	0.303
		5	2, 7-dimethyloctane	83		1.1	11.5	0.028
IV	phenol	1	propylbenzene	34.5	34.5	23.7	82.8	0.687
		2	mesitylene	33.5		18.7	74.7	0.542
		3	pseudocumene	33.0		16.2	70.1	0.470
		4	butylbenzene	35.5		7.2	45.1	0.209
V	pyrocatechol	1	1, 3, 5-triethylbenzene	40.9	40.0	31.2	87.3	0.780
		2	naphthalene	37.6		28.5	87.0	0.713
		3	2-methylnaphthalene	36.7		12.65	58.7	0.316
		4	1-methylnaphthalene	35.9		11.1	55.6	0.278
		5	biphenyl	48.0		6.05	35.5	0.151
		6	diphenylmethane	46.9		3.15	25.9	0.079
		7	acenaphtene	42.2		0.65	12.4	0.016
VI	resorcinol	1	2-methylnaphthalene	53.8	54.5	41.35	87.7	0.769
		2	1-methylnaphthalene	55.4		38.3	83.1	0.703

No.	A	No.	Homologue H_i	Z_d	Z_d'	δ_A	x_{H_i}	$\delta_{A(r)}$
VI	resorcinol	3	biphenyl	54.7	54.5	29.05	72.9	0.533
		4	diphenylmethane	53.5		22.65	65.0	0.416
		5	acenaphehene	56.9		15.2	51.7	0.279
		6	1, 2-diphenylthane	60.0		11.8	44.4	0.217
		7	fluorene	42.0		7.4	41.9	0.136
		8	stilbene	37.1		3.9	32.4	0.072
VII	methanol	1	isopropylethylene	53.4	50.0	46.65	93.5	0.933
		2	2-methylbutane	48		40.1	91.5	0.802
		3	isoprene	44		35.15	89.5	0.703
		4	*n*-pentane	46		33.85	85.5	0.677
		5	2-methyl-2-butene	46		32.95	85.0	0.659
		6	3-methyl-1, 2-butadiene	43		30.15	83.5	0.603
		7	2, 3-dimethylbutane	54		19.65	60.0	0.393
		8	biallyl	54		17.60	57	0.352
		9	*n*-hexane	59		14.15	49	0.283
		10	*n*-heptane	107		5.65	23	0.113
		11	isooctane	131		5.25	20	0.105
		12	2, 4-dimethylhexane	142		3.65	16	0.073
		13	*n*-octane	165		1.65	10	0.033
VIII	acetic acid	1	*n*-hexane	55	55.0	49.80	95	0.905
		2	*n*-heptane	87		26.33	55	0.479
		3	*n*-octane	123		12.35	32	0.225
		4	*n*-nonane	181		5.25	17	0.096
		5	*n*-decane	130		1.30	10	0.024
		6	*n*-undecane	825		0.33	.2	0.006

Summary

1. Reduced azeotropic parameters were used expressing the universal relations for series of azeotropic mixtures forming regular solutions:

(8a) $$\delta_A^{(r)} = x_{H_i}^2;$$

(9a) $$x_{H_i} = \frac{1+\Delta_A^{(r)}}{2};$$

(10a) $$\Delta_A^{(r)} = 2\sqrt{\delta_A^{(r)}} - 1.$$

The value of $\delta_A^{(r)}$ changes within the limits $0-1$ and $\Delta_A^{(r)}$ from -1 to $+1$. They represent: $\delta_A^{(r)}$ the reduced azeotropic depression in relation to the azeotropic agent A; $\Delta_A^{(r)}$ the reduced differences in the boiling temperature of agent A and the second components H_i in azeotropes of general type (AH_i); x_{H_i} – mole fraction of the considered representative H_i of the homologous series (H).

2. Six out of eight series of azeotropes with various azeotropic agents and with three series of homologues: paraffins, one ring and two ring aromatic hydrocarbons, have shown a satisfactory agreement with the theoretical prognostications.

3. In these six series showing a satisfactory agreement with the theoretically established reduced equations, acetone, *n*-propyl formate, isoamyl alcohol, phenol, pyrocatechol and resorcinol were the azeotropic agents.

4. Large deviations were found for methanol and acetic acid used as azeotropic agents.

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REFERENCES

[1] W. Świętosławski, Roczn. Chem. **10** (1930), 97.

[2] – Roczn. Chem. **25** (1951), 98, 109, 381.

[3] W. Malesiński, *Series of binary positive azeotropes*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 601.

[4] M. Lecat, *Tables Azéotropiques*, vol. I, Bruxelles 1949.

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GEOLOGY

Some Remarks on the Tectonic Structure of Caledonides and Variscides in the Sudeten

by

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A number of different rock complexes may be differentiated in the region of the Sudeten. They vary in age from the pre-Cambrian up to the Tertiary inclusive. These complexes correspond either to different diastrophic cycles, or they represent buildings of one and the same age, regionally differentiated and showing disparities in their facial development and the degree of tectonic deformations.

The principal role in the geologic structure of the Sudeten Mountains is played by rock complexes linked with the Caledonian and Variscian epoch of folding, as well as by rock groups connected with not closely defined pre-Cambrian mountain-making processes. The pre-Cambrian formations of the Sudeten Mountains, or at least rocks which may be considered as being of this age — are frequently highly metamorphosed. The Cambro-Silurian complex, particular to the Caledonides, is also nearly wholly metamorphosed. The Devonian rocks of the Variscian orogenic cycle have been metamorphosed in the Eastern Sudeten.

The stratigraphy of the Sudetic metamorphic complexes is far from being a simple one. The deductions so far reached on this subject are for the most part open to question or at least uncertain. The delimination of the Early Palaeozoic from the pre-Cambrian complex is particularly doubtful.

Another question, problematical to a large extent, concerns the stratigraphic correlations between the different regions. The difficulties in correlations are due, among other causes, to the discontinuity of the geological structure of the Sudeten. These mountains suffered a dismembering into blocks, partly during the Variscian mountain-making process, partly later, during the Tertiary. Some of these blocks were lifted and deeply eroded, others subsided in the same time, and were covered with sediments.

As a result of these processes, rock complexes of very different age came in contact with one another, these features being characteristic of the Sudeten Mountains. In other places, contact is maintained between structures of the same orogenic systems, but eroded to various tectonic levels. In regions built up of metamorphic rocks, such structures may present different mineral facies.

Uncertainties in the stratigraphy of the metamorphic rocks make the tectonical studies in the Sudeten often extremely difficult. In many cases it is impossible to determine with any accuracy the age of the deformations. Hence, the delimitation of pre-Cambrian, Caledonian and Variscian deformations is often problematic.

According to views so far established, the Caledonian orogenic system plays a major role in the western and middle part of the Sudeten, as far as the Ramsau line [1]—[4]. In the eastern Sudeten, on the other hand, the Variscian orogenic system predominates.

As mentioned above, certain rock complexes of the Sudeten belong most probably to the pre-Cambrian (western and middle Sudeten). These elements, considered as pre-Cambrian, have been mostly included into the Caledonian unities. During this process they underwent a variety of deformations and metamorphic alterations [9].

The Caledonides of the middle and western Sudeten, including the deformed remnants of pre-Cambrian elements, later underwent tectonic rebuilding in many phases, during the Variscian mountain-making epoch. The old Caledonian overthrusts were renewed during these movements [10], and a new fault pattern, new thrusts, elevations and subsidences ensued, bringing about considerable deformations on the deeply eroded Caledonian and pre-Cambrian structures. New mountains arose, built up to a great extent of old rock material.

This rebuilding of old tectonic forms was accompanied by intrusions of magma, and in the young-Variscian phases a powerful volcanism appeared, the climax of which took place during the middle part of the Lower-Permian.

It is noteworthy too, that the study of the tectonic development of the old formations in the Sudeten cannot be achieved without the examination of post-Variscian movements. Although their intensity was generally lesser, they caused great disturbances in the Caledonian and Variscian series.

The manifold deformations of the metamorphic rock complexes of the Sudeten finally produced forms of a composed tectonic character, often of an utmost complexity. It is frequently difficult to include them among any forms described in classical handbooks on structural geology. In extreme cases, ancient tectonic forms have undergone not only violent deformations, but also even dismembering into smaller and larger fragments. It was possible for the fragments of various tectonic unities to

intermingle, giving a chaotic intersection pattern, with the characteristics of a tectonic breccia, on the largest scale. The sketching of an adequate or even simply of a moderately correct geological section across the structures thus modified, is no easy task. Nor is it any easier to reconstruct the different phases of the deformations and to cover the mechanism and the sequence of the movements.

The composed tectonic structures which developed in many phases can be called polygenetic structures. They are the result of oft-repeated tectonic activity taking place under different physical conditions.

The polyphase rebuilding of pre-Cambrian and Caledonian series resulted often in polymetamorphism of the rocks involved, the kinetic agent playing the major part. Then, in some parts, mylonitisations could ensue. The processes of rolling out and crushing, the diaphtoritic changes, or the influence of magmatic intrusions — have wiped out completely, or to a large extent, the primary mineral structures and mineral complexes.

In extreme cases, the determination of the initial material of such an altered rock is very difficult, or even impossible.

The Caledonian, and subsequently the Variscian and Saxonian rebuilding did not, however, affect the whole of the basement masses to an equal degree. This is proved by the appearance of a great relict of these masses, represented in the middle Sudeten and their foreland by the large gneissic block of Sowie Góry. It is triangular in shape, and about 600 sq. km. in area (see attached map). An essential component of this block is the paragneissic complex, showing strong analogies with similar series in the Moldanubian zone of southern Bohemia. A number of arguments can be advanced in favour of the pre-Cambrian age of these paragneisses.

The Sowie Góry triangle constitutes, among the Sudetic Caledonides and Variscides, a completely strange element, which differs not only by the petrographic character of its rocks, but also by the features of its tectonic. It is also of interest that the Sudetic foldings, though of different age, are disposed approximately on lines parallel to the three sides of the triangle, formed by the block of Sowie Góry. This fact has been remarked by many authors (Cloos, Bederke, and others). In the regions of the three summits of the Sowie Góry triangle there also appear the three well known, essential virgations of the Sudetic foldings: the western virgation of Marciszów, the northern virgation of Strzelin and the southern virgation of Lądek.

The gneissic block of Sowie Góry might be considered as a part of some older pre-Cambrian orogen, which suffered denudation down to the highly metamorphosed root portions, possibly even before the Caledonian epoch of folding. This part of the ancient orogen was uplifted and has been preserved as a median mass between three plastic zones of later orogenic systems.

Three main and different directions of folding seem to have been dominant in these zones (S, SW and E).

The block of Sowie Góry was pushed over adjacent unities under the influence of extreme forces. The trend of shifting varied in time and was south, south-west or east respectively. These movements were accompanied by intrusions of magma. The block just described has been gradually deformed and diminished by breaking-up. The kinetic engagement of the border zones resulted in mylonitic transformations of the rocks involved (the south-westen border of the block and the mylonitic zone of Niemcza).

Space considerations prevent a further discussion of the questions presented in this note. The author is, however, planning to summarise in a forthcoming paper the results of new investigations so far carried out on the tectonic structure of the Sudetic Caledonides.

REFERENCES

[1] E. Bederke, *Das Devon in Schlesien und das Alter der Sudetenfaltung*, Fortschritte der Geol. u. Paläontol. (1924), No 7.

[2] — *Bau und Alter des Ostsudetischen Gebirges*, Neues Jahrbuch f. Mineralogie etc., Beil. Bd. Ser. B, **53** (1926).

[3] — *Die Grenze von Ost- und Westsudeten und ihre Bedeutung für die Einordnung der Sudeten in den Gebirgsbau Mitteleuropas*, Geol. Rundschau **20** (1929).

[4] — *Sudetenrand und Eulengneisproblem*. Vom Deutschen Osten (1931), No 21.

[5] H. Cloos, *Der Gebirgsbau Schlesiens und die Stellung seiner Bodenschätze*, Berlin, 1922.

[6] F. Dahlgrün, *Zur Altersdeutung des Vordevons im westsudetischen Schiefergebirge*, Zeitschr. d. deutsch. Geol. Gesellschaft **86** (1934).

[7] — *Bemerkungen zum kaledonischen Bau der Westsudeten*, Geotektonische Forschungen (1938), No 2.

[8] A. Polański, *On the metamorphism of crystalline formation of the Sowie Mountains (Middle Sudeten)*, Summary, Archiwum Mineralogiczne **18** (1954), No 2.

[9] K. Smulikowski, *The old crystalline formations of the Sudeten Mountains*, Annales de la Soc. géol. de Pologne **21** (1951), No 1.

[10] H. Teisseyre and K. Smulikowski, *Przekrój przez południową część Gór Kaczawskich w okolicy Bolkowa. Guide des excursions de la XXIV-e Réun. de la Soc. Géol. de Pologne dans les Sudètes en* 1951, Rocznik P. T. G. **21** (1953).

Polish-German Vocabulary of Topographic Names

Bardo — Wartha
Desna — Tess
Gierałtów — Gersdorf
Góry Bystrzyckie — Habelschwerdtergebirge
Góry Orlickie — Adlergebirge
Kaczawa — Katzbach
Kaczawa Mountains — Bober-Katzbachgebirge
Karkonosze — Riesengebirge
Kłodzko — Glatz
Lądek — Landeck
Łużyce — Lausitz
Niemcza — Nimptsch
Pradziad — Altvatergebirge
Sowie Góry — Eulengebirge
Strzegom — Striegau
Strzelin — Strehlen
Śnieżnik Kłodzki — Glatzer-Schneegebirge
Świebodzice — Freiburg
Wądroże Wielkie — Gross Wandriss,

Legend

1 Variscian granites (chiefly late-Variscian, underformed).
2 Variscian synkinematic intrusions of highly contaminated granitic magma.
3 Tonalites (Variscian, synkinematic).
4 Orthogneisses of Kepernik (Variscian synkinematic).
5 Andalusite-staurolite schists (country rocks around the Kepernik-intrusion).
6 Lower and middle-Devonian (eastern Sudeten – green schist facies).
7 Gneissic series of Desna (Caledonian?).
8 Region of Bardo-Mountains. (Culm folded with Silurian rocks of late-Caldonian belt).
9 Upper Devonian (in Culm Facies) and Lower Culm.
10 Gabbro.
11 Amphibolite.
12 Serpentinite.
13 Cambro-Silurian (green schist facies) of the Kaczawa-Caledonides. Cambro-Silurian (the same facies) on the southern slope of the Karkonosze and in the region of Łużyce.
14 Phyllites on the southern slope of the region of Góry Orlickie (pre-Cambrian?).
15 Granodiorite of Seidenberg (pre-Cambrian?).
16 Granodiorite of Łużyce (Variscian?).
17 Orthogneisses (Caledonian and pre-Cambrian?).
18 Granite of Rumburg (Caledonian or pre-Cambrian).
19 Schist complexes, partly early palaeozoic, partly pre-Cambrian. In the region of Lądek and Śnieżnik Kłodzki also the migmatic series of Gierałtów, and various mylonitic gneisses.
20 Gneissic formations of Sowie Góry, chiefly paragneisses and migmatites (early pre-Cambrian).
21 Faults and overthrusts of different orogenic phases; arrows– directions of movements.

A – granodioritic pluton of Łużyce.
B – Cambro-Silurian of Łużyce.
C–C – granitic pluton of Karkonosze.
D – Sudetic series (amphibolite facies).
E – Subsudetic series (green schist facies).
F–F – Caledonian unities of the Kaczawa-Mountains (Cambro-Silurian green schist facies).
G – gneisses of Wądroże Wielkie.
H – subsided block of Świebodzice.
I–I – middle-Sudetic trough.
f–f – the supposed southern limit of the Kaczawa Caledonides under the sedimentary mantle of the middle-Sudetic trough.
K – the region of Bardo-Mountains.
L – the cristalline series of Kłodzko – northern part.
M – the cristalline series of Kłodzko – southern part.
N – the cristalline series of the region of Lądek and of Śnieżnik Kłodzki.
O – the region of Góry Orlickie.
P – the region of Góry Bystrzyckie.
Q – the gneissic dome of Kepernik.
R – the gneissic dome of Pradziad.
S – the block of Sowie Góry.
T–T – the mylonitic zone of Niemcza.
U – the granitic pluton of Strzegom.
W – the granitic pluton of Strzelin.

BOLESŁAWIEC
LEGNICA
Odra
WROCŁAW
STRZEGOM
SOBÓTKA
JELENIA GÓRA
BOLKÓW
ŚWIEBODZICE
ŚWIDNICA
KAMIENNA GÓRA
WAŁBRZYCH
STRZELIN
TRUTNOV
BROUMOV
BARDO
HRONOV
KŁODZKO
BYSTRZYCA
ZLATE HORY
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